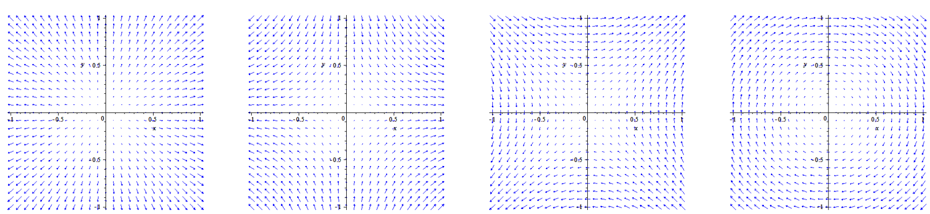
Chapter 18

The Theorems of Green, Stokes, and Gauss

Imagine a fluid or gas moving through space or on a plane. Its density may vary from point to point. Also its velocity vector may vary from point to point. Figure 18.0.1 shows four typical situations. The diagrams shows flows in the plane because it's easier to sketch and show the vectors there than in space.



(a) (b) (c) (d)

Figure 18.0.1: Four typical vector fields in the plane.

The plots in Figure 18.0.1 resemble the slope fields of Section 3.6 but now, instead of short segments, we have vectors, which may be short or long. Two questions that come to mind when looking at these vector fields:

- For a fixed region of the plane (or in space), is the amount of fluid in the region increasing or decreasing or not changing?
- At a given point, does the field create a tendency for the fluid to rotate? In other words, if we put a little propeller in the fluid would it turn? If so, in which direction, and how fast?

This chapter provides techniques for answering these questions which arise in several areas, such as fluid flow, electromagnetism, thermodynamics, and gravity. These techniques will apply more generally, to a general vector field. Applications come from magnetics as well as fluid flow.

Throughout we assume that all partial derivatives of the first and second orders exist and are continuous.

18.1 Conservative Vector Fields

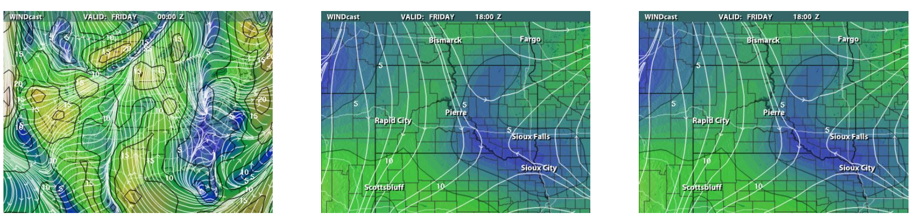
In Section 15.3 we defined integrals of the form

$$\int_C (P\ dx + Q\ dy + R\ dz). \tag{18.1.1}$$

where P, Q, and R are scalar functions of x, y, and z and C is a curve in space. Similarly, in the xy-plane, for scalar functions of x and y, P and Q, we have

$$\int_C (P\ dx + Q\ dy).$$

Instead of three scalar fields, P, Q, and R, we could think of a single vector function $\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$. Such a function is called a **vector field**, in contrast to a scalar field. It's hard to draw a vector field defined in space. However, it's easy to sketch one defined only on a plane. Figure 18.1.1 shows three wind maps, showing the direction and speed of the winds for (a) the entire United States, (b) near Pierre, SD and (c) near Tallahassee, FL on April 24, 2009.



(a) (b) (c)

Figure 18.1.1: Wind maps showing (a) a source and (b) a saddle. Obtained from `www.intellicast.com/National/Wind/Windcast.aspx` on April 23, 2009. [Another idea for these sample plots is to use maps from Hurricane Katrina.]

Introducing the formal vector $d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}$, we may rewrite (18.1.1) as

$$\int_C \mathbf{F} \cdot\ d\mathbf{r}.$$

The vector notation is compact, is the same in the plane and in space, and emphasizes the idea of a vector field. However, the clumsy notations

$$\int_C (P\,dx + Q\,dy + R\,dz) \qquad \text{and} \qquad \int_C (P(x, y, z)\,dx + Q(x, y, z)\,dy + R(x, y, z)\,dz)$$

do have two uses: to prove theorems and to carry out calculations.

Conservative Vector Fields

Recall the definition of a conservative vector field from Section 15.3.

> **DEFINITION** *(Conservative Field)* A vector field $\mathbf{F}$ defined in some planar or spatial region is called **conservative** if
>
> $$\int_{C_1} \mathbf{F} \cdot \, dr = \int_{C_2} \mathbf{F} \cdot \, dr$$
>
> whenever C_1 and C_2 are any two simple curves in the region with the same initial and terminal points.

An equivalent definition of a conservative vector field $\mathbf{F}$ is that for any simple closed curve C in the region $\oint_C \mathbf{F} \cdot \, d\mathbf{r} = 0$, as Theorem 18.1.1 implies. A closed curve is a curve that begins and ends at the same point, forming a loop. It is simple if it passes through no point — other than its start and finish points — more than once. A curve that starts at one point and ends at a different point is simple if it passes through no point more than once. Figure 18.1.2 shows some curves that are simple and some that are not.

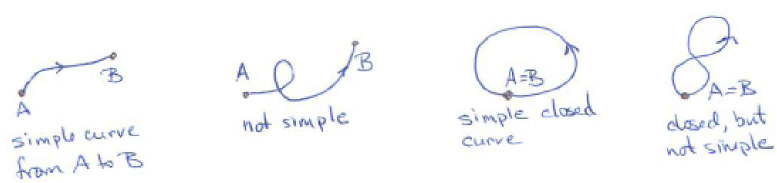


Figure 18.1.2:

Theorem 18.1.1. *A vector field* $\mathbf{F}$ *is conservative if and only if* $\oint_C \mathbf{F} \cdot \, dr = 0$ *for every simple closed curve in the region where* $\mathbf{F}$ *is defined.*

Proof

Assume that $\mathbf{F}$ is a conservative and let C be simple closed curve that starts and ends at the point A. Pick a point B on the curve and break C into two curves: C_1 from A to B and C_2^* from B to A, as indicated in Figure 18.1.3(a).

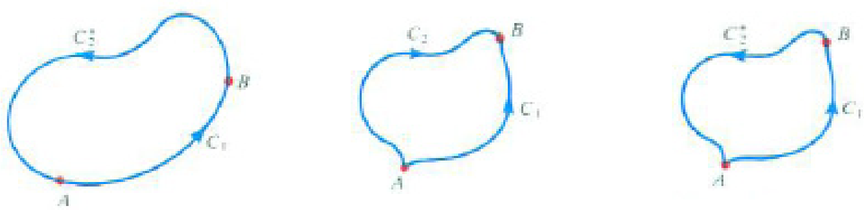
Let C_2 be the curve C_2^* traversed in the opposite direction, from A to B.

Note the sign change.

Then, since $\mathbf{F}$ is conservative,

$$\oint_C \mathbf{F} \cdot \, d\mathbf{r} \;=\; \int_{C_1} \mathbf{F} \cdot \, d\mathbf{r} + \int_{C_2^*} \mathbf{F} \, d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot \, d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot \, d\mathbf{r} = 0.$$

On the other hand, assume that $\mathbf{F}$ has the property that $\oint_C \mathbf{F}\cdot\, d\mathbf{r} = 0$ for any simple closed curve C in the region. Let C_1 and C_2 be two simple curves in the region, starting at A and ending at B. Let $-C_2$ be C_2 taken in the reverse direction. (See Figures 18.1.3(b) and (c).) Then C_1 followed by $-C_2$ is a closed curve C from A back to A. Thus



(a) (b) (c)

Figure 18.1.3:

$$0 = \oint\limits_C \mathbf{F}\cdot\, dr \;=\; \int\limits_{C_1} \mathbf{F}\cdot\, dr + \int\limits_{-C_2} \mathbf{F}\cdot\, dr = \int\limits_{C_1} \mathbf{F}\cdot\, dr - \int\limits_{C_2} \mathbf{F}\cdot\, dr.$$

Consequently,

$$\int\limits_{C_1} \mathbf{F}\cdot\, dr = \int\limits_{C_2} \mathbf{F}\cdot\, dr.$$

This concludes both directions of the argument. •

In this proof we tacitly assumed that C_1 and C_2 overlap only at their endpoints, A and B. Exercise 26 treats the case when the curves intersect elsewhere also.

Every Gradient Field is Conservative

Whether a particular vector field is conservative is important in the study of gravity, electro-magnetism, and thermodynamics. In the rest of this section we describe ways to determine whether a vector field $\mathbf{F}$ is conservative.

The first method that may come to mind is to evaluate $\oint \mathbf{F}\cdot d\mathbf{r}$ for every simple closed curve and see if it is always 0. If you find a case where it is not 0, then $\mathbf{F}$ is not conservative. Otherwise you face the task of evaluating a never-ending list of integrals checking to see if you always get 0. That is a most impractical test. Later in this section partial derivatives will be used to obtain a much simpler test. The first test involves gradients.

Gradient Fields Are Conservative

The fundamental theorem of calculus asserts that $\int_a^b f'(x)\ dx = f(b) - f(a)$. The next theorem asserts that $\int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A)$, where f is a function of two or three variables and C is a curve from A to B. Because of its resemblance to the fundamental theorem of calculus, Theorem 18.1.2 is sometimes called the **fundamental theorem of vector fields**.

Any vector field that is the gradient of a scalar field turns out to be conservative. That is the substance of Theorem 18.1.2, which says, "The circulation of a gradient field of a scalar function f along a curve is the difference in values of f at the end points."

Theorem 18.1.2. *Let f be a scalar field defined in some region in the plane or in space. Then the gradient field $\mathbf{F} = \nabla f$ is conservative. In fact, for any points A and B in the region,*

$$\int_C \nabla f \cdot\ d\mathbf{r} = f(B) - f(A).$$

Proof

For simplicity take the planar case. Let C be given by the parameterization $\mathbf{r} = \mathbf{G}(t)$ for t in $[a, b]$. Let $\mathbf{G}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$. Then,

$$\int_C \nabla f \cdot d\mathbf{r} \;=\; \int_C \left(\frac{\partial f}{\partial x}\ dx + \frac{\partial f}{\partial y}\ dy\right) = \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}\right)\ dt.$$

The integrand $(\partial f/\partial x)(dx/dt) + (\partial f/\partial y)(dy/dt)$ is reminiscent of the chain rule in Section 16.3. If we introduce the function H defined by

$$H(t) = f(x(t), y(t)),$$

then the chain rule asserts that

$$\frac{dH}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Thus

$$\int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}\right)\ dt = \int_a^b \frac{dH}{dt}\ dt = H(b) - H(a)$$

by the fundamental theorem of calculus. But

$$H(b) = f(x(b), y(b)) = f(B)$$

and

$$H(a) = f(x(a), y(a)) = f(A).$$

Consequently,

$$\int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A), \qquad (18.1.2)$$

and the theorem is proved. •

In differential form Theorem 18.1.2 reads

If f is defined as the xy-plane, and C starts at A and ends at B,

$$\int_C \left(\frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy \right) = f(B) - f(A) \qquad (18.1.3)$$

If f is defined in space, then,

$$\int_C \left(\frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy + \frac{\partial f}{\partial z}\, dz \right) = f(B) - f(A). \qquad (18.1.4)$$

Note that one vector equation (18.1.2) covers both cases (18.1.3) and (18.1.4). This illustrates an advantage of vector notation.

It is a much more pleasant task to evaluate $f(B) - f(A)$ than to compute a line integral.

EXAMPLE 1 Let $f(x, y, z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$, which is defined everywhere except at the origin. (a) Find the gradient field $\mathbf{F} = \nabla f$, (b) Compute $\int_C \mathbf{F} \cdot\, d\mathbf{r}$ where C is any curve from $(1, 2, 2)$ to $(3, 4, 0)$.

SOLUTION (a) Straightforward computations show that

$$\frac{\partial f}{\partial x} = \frac{-x}{(x^2+y^2+z^2)^{3/2}}, \frac{\partial f}{\partial y} = \frac{-y}{(x^2+y^2+z^2)^{3/2}}, \frac{\partial f}{\partial z} = \frac{-z}{(x^2+y^2+z^2)^{3/2}}.$$

So

$$\nabla f = \frac{-z\mathbf{i} - y\mathbf{j} - z\mathbf{k}}{(x^2+y^2+z^2)^{3/2}}. \qquad (18.1.5)$$

If we let $\mathbf{r}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $r = \|\mathbf{r}\|$, and $\widehat{\mathbf{r}} = \mathbf{r}/r$, then (18.1.5) can be written more simply as

$$\mathbf{F} = \nabla f = \frac{-\mathbf{r}}{r^3} = \frac{-\widehat{\mathbf{r}}}{r^2}.$$

(b) For any curve C from $(1,2,2)$ to $(3,4,0)$,

$$\begin{aligned}\int_C \nabla f \cdot \, d\mathbf{r} &= f(3,4,0) - f(1,2,2) = \frac{1}{\sqrt{3^2+4^2+0^2}} - \frac{1}{\sqrt{1^2+2^2+2^2}} \\ &= \frac{1}{5} - \frac{1}{3} = -\frac{2}{15}.\end{aligned}$$

$\diamond$

For a constant k, positive or negative, any vector field, $\mathbf{F} = k\widehat{\mathbf{r}}/r^2$, is called an **inverse square central field**. They play an important role in the study of gravity and electromagnetism.

In Example 1 $\|\nabla f\| = \frac{\|-\mathbf{r}\|}{r^3} = \frac{r}{r^3} = \frac{1}{r^2}$ and $f(x,y,z) = \frac{1}{r}$. In the study of gravity, ∇f measures gravitational attraction, and f measures "potential."

EXAMPLE 2 Evaluate $\oint_C (y \; dx + x \; dy)$ around a closed curve C taken counterclockwise.

SOLUTION In Section 15.3 it was shown that if the area enclosed by a curve C is A, then $\oint_C x \; dy = A$ and $\oint_C y \; dx = -A$. Thus,

$$\oint_C (y \; dx + x \; dy) = -A + A = 0.$$

A second solution uses Theorem 18.1.2. Note that

$$\nabla(xy) = \frac{\partial(xy)}{\partial x}\mathbf{i} + \frac{\partial(xy)}{\partial y}\mathbf{j} = y\mathbf{i} + x\mathbf{j},$$

that is, the gradient of xy is $y\mathbf{i} + x\mathbf{j}$.

Hence, byTheorem 18.1.2, if the endpoints of C are A and B

$$\oint_C (y \; dx + x \; dy) = \oint_C \nabla(xy) \cdot d\mathbf{r} = xy|_A^B .$$

Because C is a closed curve, $A = B$ and so the integral is 0. $\diamond$

A differential form $P(x,y,z)\; dx + Q(x,y,z)\; dy + R(x,y,z)\; dz$ is called **exact** if there is a scalar function f such that $P(x,y,z) = \partial f/\partial x$, $Q(x,y,z) = \partial f/\partial y$, and $R(x,y,z) = \partial f/\partial z$. In that case, the expression takes the form

$$\frac{\partial f}{\partial x}\; dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z}\; dz.$$

That is the same thing as saying that the vector field $\mathbf{F} = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$ is a gradient field: $\mathbf{F} = \nabla f$.

If F is Conservative Must It Be a Gradient Field?

The proof of the next theorem is similar to the proof of the second part of the Fundamental Theorem of Calculus. We suggest you review that proof (page 469) before reading the following proof.

FTC II states that every continuous function has an antiderivative.

The question may come to mind, *"If* $\mathbf{F}$ *is conservative, is it necessarily the gradient of some scalar function?"* The answer is "yes." That is the substance of the next theorem, but first we need to introduce some terminology about regions.

A region $\mathcal{R}$ in the plane is **open** if for each point P in $\mathcal{R}$ there is a disk with center at P that lies entirely in $\mathcal{R}$. For instance, a square *without its edges* is open. However, a square *with its edges is not open.*

An open region in space is defined similarly, with "disk" replaced by "ball."

An open region $\mathcal{R}$ is **arcwise-connected** if any two points in it can be joined by a curve that lies completely in $\mathcal{R}$. In other words, it consists of just one piece.

Theorem 18.1.3. *Let* $\mathbf{F}$ *be a conservative vector field defined in some arcwise-connected region* $\mathcal{R}$ *in the plane (or in space). Then there is a scalar function* f *defined in that region such that* $\mathbf{F} = \nabla f$.

Proof

Consider the case when $\mathbf{F}$ is planar, $\mathbf{F} = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$. (The case where $\mathbf{F}$ is defined in space is similar.) Define a scalar function f as follows. Let (a,b) be a fixed point in $\mathcal{R}$ and (x,y) be any point in $\mathcal{R}$. Select a curve C in $\mathcal{R}$ that starts at (a,b) and ends at (x,y).

Define $f(x,y)$ to be $\int_C \mathbf{F}\cdot\, d\mathbf{r}$. Since $\mathbf{F}$ is conservative, the number $f(x,y)$ depends only on the point (x,y) and not on the choice of C. (See Figure 18.1.4.)

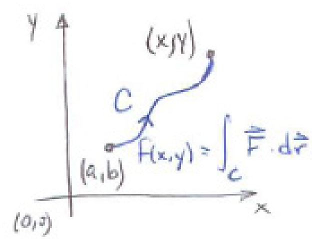


Figure 18.1.4:

All that remains is to show that $\nabla f = \mathbf{F}$; that is, $\partial f/\partial x = P$ and $\partial f/\partial y = Q$. We will go through the details for the first case, $\partial f/\partial x = P$. The reasoning for the other partial derivative is similar.

Let (x_0, y_0) be an arbitrary point in $\mathcal{R}$ and consider the difference quotient whose limit is $\partial f/\partial x(x_0,y_0)$, namely,

$$\frac{f(x_0+h,y_0) - f(x_0,y_0)}{h},$$

for h small enough so that (x_0+h, y_0) is also in the region.

Let C_1 be any curve from (a,b) to (x_0,y_0) and let C_2 be the straight path from (x_0,y_0) to (x_0+h,y_0). (See Figure 18.1.5.) Let C be the curve from $(0,0)$ to the point(x_0+h,y_0) formed by taking C_1 first and continuing on C_2. Then

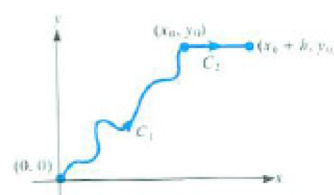


Figure 18.1.5:

$$f(x_0,y_0) = \int\limits_{C_1} \mathbf{F}\cdot d\mathbf{r},$$

and

$$f(x_0+h,y_0)=\int\limits_C \mathbf{F}\cdot d\mathbf{r}=\int\limits_{C_1}\mathbf{F}\cdot d\mathbf{r}+\int\limits_{C_2}\mathbf{F}\cdot d\mathbf{r}.$$

Thus

$$\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}=\frac{\int_{C_2}\mathbf{F}\cdot d\mathbf{r}}{h}=\frac{\int_{C_2}(P(x,y)\ dx+Q(x,y)\ dy)}{h}.$$

On C_2, y is constant, $y=y_0$; hence $dy=0$. Thus $\int_{C_2}Q(x,y)\ dy=0$. Also,

$$\int\limits_{C_2}P(x,y)\ dx=\int\limits_x^{x+h}P(x,y)\ dx.$$

See Section 6.3 for the MVT for Definite Integrals

By the Mean-Value Theorem for definite integrals, there is a number x^* between x and $x+h$ such that

$$\int\limits_x^{x+h}P(x,y)\ dx=P(x^*,y_0)h.$$

Hence

$$\begin{aligned}\frac{\partial f}{\partial x}(x_0,y_0) &= \lim_{h\to 0}\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}\\ &= \lim_{h\to 0}\frac{1}{h}\int\limits_{x_0}^{x_0+h}P(x,y_0)\ dx=\lim_{h\to 0}P(x^*,y_0)=P(x_0,y_0).\end{aligned}$$

Consequently,

$$\frac{\partial f}{\partial x}(x_0,y_0)=P(x_0,y_0),$$

as was to be shown.

In a similar manner, we can show that

$$\frac{\partial f}{\partial y}(x_0,y_0)=Q(x_0,y_0).$$

•

For a vector field $\mathbf{F}$ defined throughout some region in the plane (or space) the following three properties are therefore equivalent: Figure 18.1.6 tells us that any one of the three properties, (1), (2), or (3), describes a conservative field. We used property (3) as the definition.

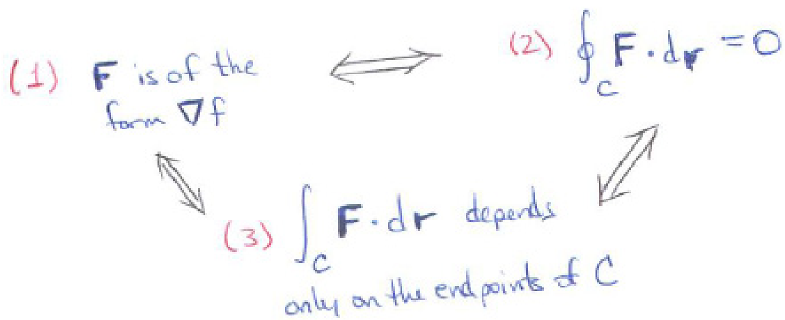


Figure 18.1.6: Double-headed arrows ($\Leftrightarrow$) mean "if and only if" or "is equivalent to." (Single-headed arrows ($\Rightarrow$) mean "implies.")

Almost A Test For Being Conservative

Figure 18.1.6 describes three ways of deciding whether a vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is conservative. Now we give a simple way to tell that it is *not* conservative. The method is simpler than finding a particular line integral $\int_C \mathbf{F} \cdot \, d\mathbf{r}$ that is not 0.

Remember that we have assumed that all of the functions we encounter in this chapter have continuous first and second partial derivatives.

The test depends on the fact that the two orders in which are may compute a second-order mixed partial derivative give the same result. (We used this fact in Section 16.8 in a thermodynamics context.)

Consider an expression of the form $P\ dx + Q\ dy + R\ dz$ (or equivalently a vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$). If the form is exact, then $\mathbf{F}$ is a gradient and there is a scalar function f such that

$$\frac{\partial f}{\partial x} = P, \qquad \frac{\partial f}{\partial y} = Q, \qquad \frac{\partial f}{\partial z} = R.$$

Since

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right),$$

we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Similarly we find

$$\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y} \qquad \text{and} \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}.$$

To summarize,

If the vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is conservative, then

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0, \qquad \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} = 0, \qquad \frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} = 0. \tag{18.1.6}$$

If at least one of these three equations (18.1.6) doesn't hold, then $P\ dx + Q\ dy + R\ dz$ is *not* exact (and $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is not conservative).

EXAMPLE 3 Show that $\cos(y)\ dx + \sin(xy)\ dy + \ln(1+x)\ dz$ is *not* exact.
SOLUTION Checking whether the first equation in (18.1.6) holds we compute

$$\frac{\partial(\sin(xy))}{\partial x} - \frac{\partial(\cos(y))}{\partial y},$$

which equals

$$y\cos(xy) + \sin(y),$$

which is not 0. There's no need to check the remaining two equations in (18.1.6). The expression $\sin(xy)\ dx + \cos(y)\ dy + \ln(1+x)\ dz$ is not exact. (Equivalently, the vector field $\sin(xy)\mathbf{i} + \cos(y)\mathbf{j} + \ln(1+x)\mathbf{k}$ is not a gradient field, hence not conservative.) ⋄

Notice that we completed Example 3 without doing any integration.

We can restate the three equations (18.1.6) as a single vector equation, by introducing a 3 by 3 formal determinant

$$\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{pmatrix} \tag{18.1.7}$$

Expanding this as though the nine entries were numbers, we get

$$\mathbf{i}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) - \mathbf{j}\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + \mathbf{k}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right). \tag{18.1.8}$$

If the three scalar equations in (18.1.6) hold, then (18.1.8) is the $\mathbf{0}$-vector. In view of the importance of the vector (18.1.8), it is given a name.

DEFINITION *(Curl of a Vector Field)* The **curl** of the vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is the vector field given by the formula (18.1.7) or (18.1.8). It is denoted $\mathbf{curl}\,\mathbf{F}$.

The formal determinant (18.1.7) is like the one for the cross product of two vectors. For this reason, it is also denoted $\nabla \times \mathbf{F}$ (read as "del cross F"). That's a lot easier to write than (18.1.8), which refers to the components. Once again we see the advantage of vector notation.

The definition also applies to a vector field $\mathbf{F} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ in the plane. Writing $\mathbf{F}$ as $P(x, y)\mathbf{i} + Q(x, y)\mathbf{j} + 0\mathbf{k}$ and observing that $\partial Q/\partial z = 0$ and $\partial P/\partial z = 0$, we find that

$$\nabla \times \mathbf{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}.$$

EXAMPLE 4 Compute the curl of $\mathbf{F} = xyz\mathbf{i} + x^2\mathbf{j} - xy\mathbf{k}$.
SOLUTION The curl of $\mathbf{F}$ is given by

$$\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & x^2 & -xy, \end{pmatrix}$$

which is short for

$$\begin{aligned} &\left(\frac{\partial}{\partial y}(-xy) - \frac{\partial}{\partial z}(x^2)\right)\mathbf{i} - \left(\frac{\partial}{\partial x}(-xy) - \frac{\partial}{\partial z}(xyz)\right)\mathbf{j} + \left(\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xyz)\right)\mathbf{k} \\ &= (-x - 0)\mathbf{i} - (-y - xy)\mathbf{j} + (2x - xz)\mathbf{k} \\ &= -x\mathbf{i} + (y + xy)\mathbf{j} + (2x - xz)\mathbf{k}. \end{aligned}$$

◇

If any case, in view of (18.1.6), for vector fields in space or in the xy-plane we have this theorem.

Theorem 18.1.4. *If* $\mathbf{F}$ *is a conservative vector field, then* $\nabla \times \mathbf{F} = \mathbf{0}$.

You may wonder why the vector field **curl F** obtained from the vector field $\mathbf{F}$ is called the "curl of $\mathbf{F}$." Here we came upon the concept purely mathematically, but, as you will see in Section 18.6 it has a physical significance: If $\mathbf{F}$ describes a fluid flow, the curl of $\mathbf{F}$ describes the tendency of the fluid to rotate and form whirlpools — in short, to "curl."

The Converse of Theorem 18.1.4 Isn't True

Warning: The converse of Theorem 18.1.4 is false.

It would be delightful if the converse of Theorem 18.1.4 were true. Unfortunately, it is not. There are vector fields $\mathbf{F}$ whose curls are $\mathbf{0}$ that are not conservative. Example 5 provides one such $\mathbf{F}$ in the xy-plane. Its curl is $\mathbf{0}$ but it is not conservative, that is, $\nabla \times \mathbf{F} = \mathbf{0}$ and there is a closed curve C with $\oint_C \mathbf{F} \cdot d\mathbf{r}$ not zero.

EXAMPLE 5 Let $\mathbf{F} = \frac{-y\mathbf{i}}{x^2+y^2} + \frac{x\mathbf{j}}{x^2+y^2}$. Show that (a) $\nabla \times \mathbf{F} = \mathbf{0}$, but (b) $\mathbf{F}$ is not conservative.

SOLUTION (a) We must compute

$$\det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{pmatrix}$$

which equals

$$\left(\frac{\partial (0)}{\partial y} - \frac{\partial}{\partial z}\left(\frac{x}{x^2+y^2} \right) \right) \mathbf{i} - \left(\frac{\partial (0)}{\partial x} - \frac{\partial}{\partial z}\left(\frac{-y}{x^2+y^2} \right) \right) \mathbf{j} + \left(\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2} \right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2} \right) \right) \mathbf{k}.$$

The $\mathbf{i}$ and $\mathbf{j}$ components are clearly 0, and a direct computation shows that the $\mathbf{k}$ component is

$$\frac{y^2 - x^2}{(x^2+y^2)^2} - \frac{y^2 - x^2}{(x^2+y^2)^2} = 0.$$

Thus the curl of $\mathbf{F}$ is $\mathbf{0}$.

(b) To show that $\mathbf{F}$ is *not* conservative, it suffices to exhibit a closed curve C such that $\oint_C \mathbf{F} \cdot d\mathbf{r}$ is not 0. One such choice for C is the unit circle parameterized counterclockwise by

$$x = \cos(\theta), \qquad y = \sin(\theta), \qquad 0 \le \theta \le 2\pi.$$

On this curve $x^2 + y^2 = 1$. Figure 18.1.7 shows a few values of $\mathbf{F}$ at points on C. Clearly $\int_C \mathbf{F} \cdot d\mathbf{r}$, which measures circulation, is positive, not 0. However, if you have any doubt, here is the computation of $\int_C \mathbf{F} \cdot d\mathbf{r}$:

Recall that, on C, $x^2 + y^2 = 1$.

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \oint_C \left(\frac{-y\, dx}{x^2+y^2} + \frac{x\, dy}{x^2+y^2} \right) \\ &= \int_0^{2\pi} (-\sin\theta\ d(\cos\theta) + \cos\theta\ d(\sin\theta)) \\ &= \int_0^{2\pi} (\sin^2\theta + \cos^2\theta)\ d\theta = \int_0^{2\pi} d\theta = 2\pi. \end{aligned}$$

This establishes (b), $\mathbf{F}$ is not conservative. ◇

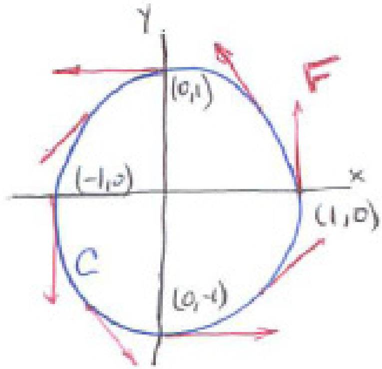


Figure 18.1.7:

The curl of $\mathbf{F}$ being $\mathbf{0}$ is not enough to assure us that a vector field $\mathbf{F}$ is conservative. An extra condition must be satisfied by $\mathbf{F}$. This condition concerns the domain of $\mathbf{F}$. This extra assumption will be developed for planar fields in Section 18.2 and for spatial fields $\mathbf{F}$ in Section 18.6. Then we will have a simple test for determining whether a vector field is conservative.

Summary

We showed that a vector field being conservative is equivalent to its being the gradient of a scalar field. Then we defined the curl of a vector field. If a field is denoted $\mathbf{F}$, the curl of $\mathbf{F}$ is a new vector field denoted $\mathbf{curl}\,\mathbf{F}$ or $\nabla \times \mathbf{F}$. If $\mathbf{F}$ is conservative, then $\nabla \times \mathbf{F}$ is $\mathbf{0}$. However, if the curl of $\mathbf{F}$ is $\mathbf{0}$, it does not follow that $\mathbf{F}$ is conservative. An extra assumption (on the domain of $\mathbf{F}$) must be added. That assumption will be described in the next section.

EXERCISES for Section 18.1 *Key:* R–routine, M–moderate, C–challenging

In Exercises 1 to 4 answer "True" or "False" and explain.

1.[R] "If $\mathbf{F}$ is conservative, then $\nabla \times \mathbf{F} = \mathbf{0}$."

2.[R] "If $\nabla \times \mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is conservative."

3.[R] "If $\mathbf{F}$ is a gradient field, then $\nabla \times \mathbf{F} = \mathbf{0}$."

4.[R] "If $\nabla \times \mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is a gradient field."

5.[R] Using information in this section, describe various ways of showing a vector field $\mathbf{F}$ is *not* conservative.

6.[R] Using information in this section, describe various ways of showing a vector field $\mathbf{F}$ *is* conservative.

7.[R] Decide if each of the following sets is open, closed, neither open nor closed, or both open and closed.

(a) unit disk with its boundary

(b) unit disk without any of its boundary points

(c) the x-axis

(d) the entire xy-plane

(e) the xy-plane with the x-axis removed

(f) a square with all four of its edges (and corners)

(g) a square with all four of its edges but with its corners removed

(h) a square with none of its edges (and corners)

8.[R] In Example 1 we computed a certain line integral by using the fact that the vector field $(-x\mathbf{i} - y\mathbf{j} - z\mathbf{k})/(x^2 + y^2 + z^2)^{3/2}$ is a gradient field. Compute that integral directly, without using the information that the field is a gradient.

9.[R] Let $\mathbf{F} = y\cos(x)\mathbf{i} + (\sin(x) + 2y)\mathbf{j}$.

(a) Show that $\mathbf{curl}\,\mathbf{F}$ is $\mathbf{0}$ and $\mathbf{F}$ is defined in an arcwise-connected region of the plane.

(b) Construct a "potential function" f whose gradient is $\mathbf{F}$.

10.[R] Let $f(x, y, z) = e^{3x}\ln(z + y^2)$. Compute $\int_C \nabla f \cdot\, d\mathbf{r}$, where C is the straight path from $(1, 1, 1)$ to $(4, 3, 1)$.

11.[R] We obtained the first of the three equations in (18.1.6). Derive the other two.

12.[R] Find the curl of $\mathbf{F}(x, y, z) = e^{x^2}yz\mathbf{i} + x^3\cos^2 3y\mathbf{j} + (1 + x^6)\mathbf{k}$.

13.[R] Find the curl of $\mathbf{F}(x, y) = \tan^2(3x)\mathbf{i} + e^{3x}\ln(1 + x^2)\mathbf{j}$.

14.[R] Using theorems of this section, explain why the curl of a gradient is $\mathbf{0}$, that is, $\mathbf{curl}(\nabla f) = \mathbf{0}$ ($\nabla \times \nabla f = \mathbf{0}$) for a scalar function $f(x, y, z)$. HINT: No computations are needed.

15.[R] By a computation using components, show that for the scalar function $f(x, y, z)$, $\mathbf{curl}\,\nabla f = \mathbf{0}$.

16.[R] Let $f(x, y) = \cos(x + y)$. Evaluate $\int_C \nabla f \cdot\, d\mathbf{r}$, where C is the curve that lies on the parabola $y = x^2$ and goes from $(0, 0)$ to $(2, 4)$.

17.[R] In Example 5 we computed $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \frac{-y\mathbf{i}+x\mathbf{j}}{x^2+y^2}$ and C is the unit circle with center at the origin. Compute the integral when C is the circle of radius 5 with center at the origin.

18.[M] In Example 5 we computed $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = \frac{-y\mathbf{i}+x\mathbf{j}}{x^2+y^2}$ and C is the unit circle with center at the origin.

(a) *Without doing any new computations*, evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where C is the square path with vertices $(1, 0)$, $(2, 0)$, $(2, 1)$, $(1, 1)$, $(1, 0)$.

(b) Evaluate the integral in (a) by a direct computation, breaking the integral into four integrals, one over each edge.

19.[M] If $\mathbf{F}$ and $\mathbf{G}$ are conservative, is $\mathbf{F}+\mathbf{G}$?

20.[M] By a direct computation, show that $\mathbf{curl}(f\mathbf{F}) = \nabla f \times \mathbf{F} + f\,\mathbf{curl}\,\mathbf{F}$.

21.[M] By a direct computation, show that $\mathbf{curl}(\mathbf{F}\times\mathbf{G}) = (\mathbf{G}\cdot\nabla)\mathbf{F} - (\mathbf{F}\cdot\nabla)\mathbf{G} + \mathbf{F}(\nabla\cdot\mathbf{G}) - \mathbf{G}(\nabla\cdot\mathbf{F})$. Each of the first two terms has a form not seen before now in this text. Here is how to interpret them when $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ and $\mathbf{G} = G_1\mathbf{i} + G_2\mathbf{j} + G_3\mathbf{k}$:

$$(\mathbf{G}\cdot\nabla)\mathbf{F} = G_1\frac{\partial F_1}{\partial x} + G_2\frac{\partial F_2}{\partial y} + G_3\frac{\partial F_3}{\partial z}.$$

22.[M] If $\mathbf{F}$ and $\mathbf{G}$ are conservative, is $\mathbf{F}\times\mathbf{G}$?

23.[M] Explain why the curl of a gradient field is the zero vector, that is, $\nabla\times\nabla f = \mathbf{0}$.

24.[M] Assume that $\mathbf{F}(x,y)$ is conservative. Let C_1 be the straight path from $(0,0,0)$ to $(1,0,0)$, C_2 the straight path from $(1,0,0)$ to $(1,1,1)$. If $\int_{C_1}\mathbf{F}\,d\mathbf{r} = 3$ and $\int_{C_2}\mathbf{F}\,d\mathbf{r} = 4$, what can be said about $\int_C \mathbf{F}\,d\mathbf{r}$, where C is the straight path from $(0,0,0)$ to $(1,1,1)$?

25.[M] Let $\mathbf{F}(x,y)$ be a field that can be written in the form

$$\mathbf{F}(x,y) = g(\sqrt{x^2+y^2})\frac{x\mathbf{i}+y\mathbf{j}}{\sqrt{x^2+y^2}}$$

where g is a scalar function. If we denote $x\mathbf{i}+y\mathbf{j}$ as $\mathbf{r}$, then $\mathbf{F}(x,y) = g(r)\widehat{\mathbf{r}}$, where $r = \|\mathbf{r}\|$ and $\widehat{\mathbf{r}} = \|\mathbf{r}\|/r$. Show that $\oint_C \mathbf{F}\cdot\,d\mathbf{r} = 0$, for any path $ABCDA$ of the form shown in Figure 18.1.8. (The path consists of two circular arcs and parts of two rays from the origin.)

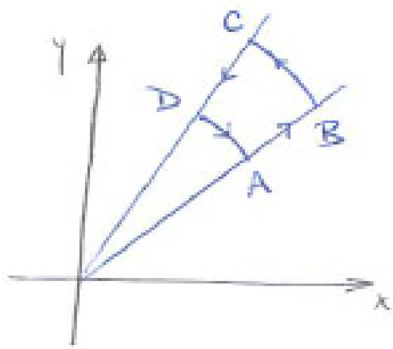


Figure 18.1.8:

26.[M] In Theorem 18.1.1 we proved that $\partial f/\partial x = P$. Prove that $\partial f/\partial y = Q$.

27.[C] In view of the previous exercise, we may expect $\mathbf{F}(x,y) = g(\sqrt{x^2+y^2})\frac{x\mathbf{i}+y\mathbf{j}}{\sqrt{x^2+y^2}}$ to be conservative. Show that it is by showing that $\mathbf{F}$ is the gradient of $G(x,y) = H(\sqrt{x^2+y^2})$, where H is an antiderivative of g, that is, $H' = g$.

28.[C] The domain of a vector field $\mathbf{F}$ is all of the xy-plane. Assume that there are two points A and B such that $\int_C \mathbf{F}\,d\mathbf{r}$ is the same for all curves C from A to B. Deduce that $\mathbf{F}$ is conservative.

29.[C] A gas at temperature T_0 and pressure P_0 is brought to the temperature $T_1 > T_0$ and pressure $P_1 > P_0$. The work done in this process is given by the line integral in the TP- plane

$$\int_C \left(\frac{RT\,dP}{P} - R\,dT\right),$$

where R is a constant and C is the curve that records the various combinations of T and P during the process. Evaluate this integral over the following paths, shown in Figure 18.1.9.

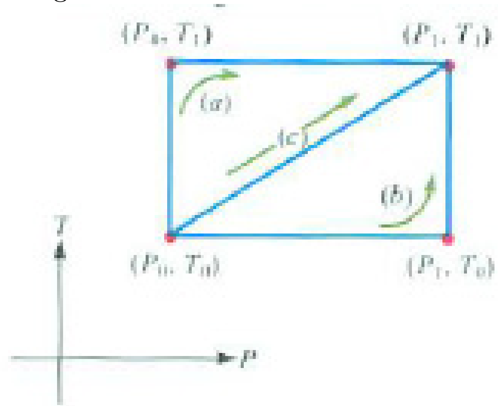


Figure 18.1.9:

(a) The pressure is kept constant at P_0 while the temperature is raised from T_0 to T_1; then the temperature is kept constant at T_1 while the pressure is raised from P_0 to P_1.

(b) The temperature is kept constant at T_0 while the pressure is raised from P_0 to P_1; then the temperature is raised from T_0 to T_1 while the pressure is kept constant at P_1.

(c) Both pressure and temperature are raised simultaneously in such a way that the path from (P_0, T_0) to (P_1, T_1) is straigh

Because the integrals are path de
ential expression $RT\ dP/P - R\ d$
dynamic quantity that depends o
just on the state. Vectorially speak
$(RT/P)\mathbf{i} - R\mathbf{j}$ is not conservative.

30.[C] Assume that $\mathbf{F}(x, y)$ is defi
xy-plane and that $\oint_C \mathbf{F}(x, y) \cdot d\mathbf{r}$ =
curve that can fit inside a disk of d
that $\mathbf{F}$ is conservative.

31.[C] This exercise completes
rem 18.1.1 in the case when C_1 and
of their endpoints A and B. In th
third simple curve from A to B that
only at A and B. Then an argume
the proof of Theorem 18.1.1 can dis

32.[C] We proved that lim $\frac{\int_{x_0}^{x}}{}$
$P(x_0, y_0)$, by using the Mean Valu
nite integrals. Find a different proo
uses a part of the Fundamental The

18.2 Green's Theorem and Circulation

In this section we discuss a theorem that relates an integral of a vector field over a closed curve C in a plane to an integral of a related scalar function over the region $\mathcal{R}$ whose boundary is C. We will also see what this means in terms of the circulation of a vector field.

Statement of Green's Theorem

We begin by stating Green's Theorem and explaining each term in it. Then we will see several applications of the theorem. Its proof is at the end of the next section.

There are two analogs of Green's Theorem in space; they are discussed in Sections 18.5 and 18.6.

Green's Theorem

Let C be a simple, closed counterclockwise curve in the xy-plane, bounding a region $\mathcal{R}$. Let P and Q be scalar functions defined at least on an open set containing $\mathcal{R}$. Assume P and Q have continuous first partial derivatives. Then

$$\oint_C (P\ dx + Q\ dy) = \int_{\mathcal{R}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ dA.$$

Recall, from Section 18.1, that a curve is closed when it starts and ends at the same point. It's simple when it does not intersect itself (except at its start and end). These restrictions on C ensure that it is the boundary of a region $\mathcal{R}$ in the xy-plane.

Since P and Q are independent of each other, Green's Theorem really consists of two theorems:

$$\int_C P\ dx = -\int_R \frac{\partial P}{\partial y}\ dA \qquad \text{and} \qquad \oint_C Q\ dy = \int_R \frac{\partial Q}{\partial x}\ dA. \tag{18.2.1}$$

EXAMPLE 1 In Section 15.3 we showed that if the counterclockwise curve C bounds a region $\mathcal{R}$, then $\oint_C y\ dx$ is the negative of the area of $\mathcal{R}$. Obtain this result with the aid of Green's Theorem.

SOLUTION Let $P(x,y) = y$, and $Q(x,y) = 0$. Then Green's Theorem says that

$$\oint_C y\ dx = -\int_R \frac{\partial y}{\partial y}\ dA.$$

Since $\partial y/\partial y = 1$, it follows that $\oint y\ dx$ is $-\int_{\mathcal{R}} 1\ dA$, the negative of the area of $\mathcal{R}$. ◇

Green's Theorem and Circulation

What does Green's Theorem say about a vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$? First of all, $\oint_C (P\ dx + Q\ dy)$ now becomes simply $\oint_C \mathbf{F} \cdot\ d\mathbf{r}$.

The right hand side of Green's Theorem looks a bit like the curl of a vector field in the plane. To be specific, we compute the curl of $\mathbf{F}$:

$$\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ P(x,y) & Q(x,y) & 0 \end{pmatrix} = 0\mathbf{i} - 0\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

Thus the curl of $\mathbf{F}$ equals the vector function

$$\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}. \tag{18.2.2}$$

To obtain the (scalar) integrand on the right-hand side of (18.2.2), we "dot (18.2.2) with $\mathbf{k}$,"

$$\left(\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}\right)\cdot\mathbf{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}.$$

Green's Theorem Expressed in Terms of Circulation

We can now express Green's Theorem using vectors. In particular, circulation around a closed curve can be expressed in terms of a double integral of the curl over a region.

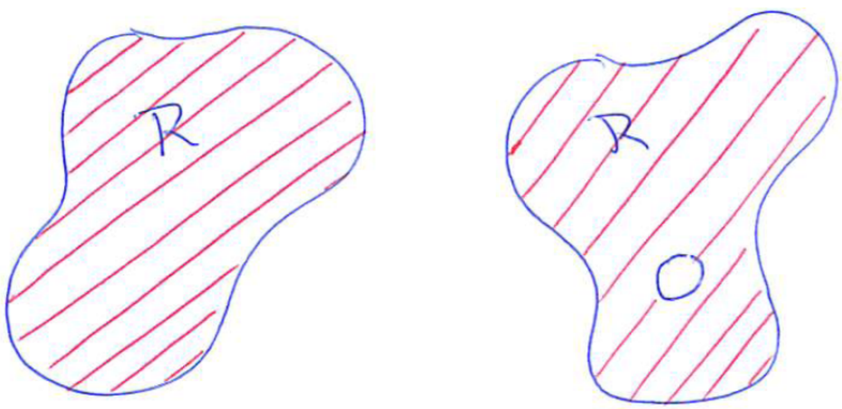
If the counterclockwise closed curve C bounds the region $\mathcal{R}$, then

$$\oint_C \mathbf{F} \cdot\ d\mathbf{r} = \int_{\mathcal{R}} (\nabla \times \mathbf{F}) \cdot \mathbf{k}\ dA.$$

Recall that if $\mathbf{F}$ describes the flow of a fluid in the xy-plane, then $\oint_C \mathbf{F} \cdot d\mathbf{r}$ represents its circulation, or tendency to form whirlpools. This theorem tells us that the magnitude of the curl of $\mathbf{F}$ represents the tendency of the fluid to rotate. If the curl of $\mathbf{F}$ is $\mathbf{0}$ everywhere, then $\mathbf{F}$ is called **irrotational** — there is no rotational tendency.

This form of Green's theorem provides an easy way to show that a vector field $\mathbf{F}$ is conservative. It uses the idea of a simply-connected region. Informally "a simply-connected region in the xy-plane comes in one piece and has no holes." More precisely, an arcwise-connected region $\mathcal{R}$ in the plane or in space is **simply-connected** if each closed curve in $\mathcal{R}$ can be shrunk gradually to a point while remaining in $\mathcal{R}$.

Figure 18.2.1 shows two regions in the plane. The one on the left is simply-connected, while the one on the right is not simply connected. For instance, the



(a) (b)

Figure 18.2.1: Regions in the plane that are (a) simply connected and (b) not simply connected.

xy-plane is simply connected. So is the xy-plane without its positive x-axis. However, the xy-plane, without the origin is *not* simply connected, because a circular path around the origin cannot be shrunk to a point while staying within the region.

If the origin is removed from xyz-space, what is left *is* simply connected. However, if we remove the z-axis, what is left is *not* simply connected.

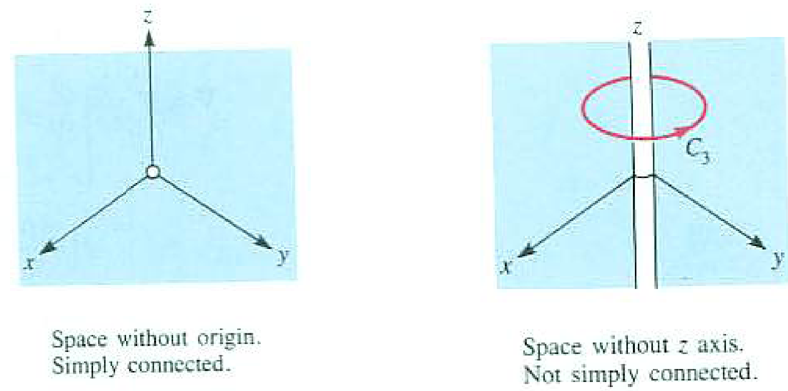
Figure 18.2.2(b) shows a curve that cannot be shrunk to a point while avoiding the z-axis.

Now we can state an easy way to tell whether a vector field is conservative.

Theorem. *If a vector field* $\mathbf{F}$ *is defined in a simply-connected region in the* xy*-plane and* $\nabla \times \mathbf{F} = \mathbf{0}$ *throughout that region, then* $\mathbf{F}$ *is conservative.*

Proof

Let C be any simple closed curve in the region and $\mathcal{R}$ the region it bounds.



(a) (b)

Figure 18.2.2: (a) xyz-space with the origin removed is simply connected. (b) xyz-space with the z-axis removed is not simply connected.

We wish to prove that the circulation of **F** around C is **0**. We have

$$\oint_C \mathbf{F}\cdot\, d\mathbf{r} = \int_{\mathcal{R}} (\mathbf{curl}\,\mathbf{F})\cdot\mathbf{k}\; dA.$$

Since **curl F** is **0** throughout $\mathcal{R}$, it follows that $\oint_C \mathbf{F}\cdot\, d\mathbf{r} = 0$. •

In Example 5 in Section 18.1, there is a vector field whose curl is **0** but is not conservative. In view of the theorem just proved, its domain must not be simply connected. Indeed, the domain of the vector field in that example is the xy-plane with the origin deleted.

EXAMPLE 2 Let $\mathbf{F}(x,y,z) = e^x y\mathbf{i} + (e^x + 2y)\mathbf{j}$.

1. Show that **F** is conservative.

2. Exhibit a scalar function f whose gradient is **F**.

SOLUTION

1. A straightforward calculation shows that $\nabla\times\mathbf{F} = \mathbf{0}$. Since **F** is defined throughout the xy-plane, a simply-connected region, Theorem 18.2 tells us that **F** is conservative.

2. By Section 18.1, we know that there is a scalar function f such that $\nabla f = \mathbf{F}$. There are several ways to find f. We show one of these methods here. Additional approaches are pursued in Exercises 7 and 8.

The approach chosen here follows the construction in the proof of Theorem 18.1.3. For a point (a, b), define $f(a, b)$ to equal $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is any curve from $(0, 0)$ to (a, b). Any curve with the prescribed endpoints will do. For simplicity, choose C to be the curve that goes from $(0, 0)$ to (a, b) in a straight line. (See Figure 18.2.3.) When a is not zero, we can use x as a parameter and write this segment as: $x = t$, $y = (b/a)t$ for $0 \leq t \leq a$. (If $a = 0$, we would use y as a parameter.) Then

$$\begin{aligned} f(a,b) &= \int_C (e^x y\,dx + (e^x + 2y)\,dy) = \int_0^a \left(e^t \frac{b}{a} t\,dt + \left(e^t + 2\frac{b}{a} t \right) \frac{b}{a}\,dt \right) \\ &= \frac{b}{a} \int_0^a \left(te^t + e^t + 2\frac{b}{a} t \right)\,dt = \frac{b}{a} \left((t-1)e^t + e^t + \frac{b}{a} t^2 \right) \Big|_0^a \\ &= \frac{b}{a} \left(te^t + \frac{b}{a} t^2 \right) \Big|_0^a = be^a + b^2. \end{aligned}$$

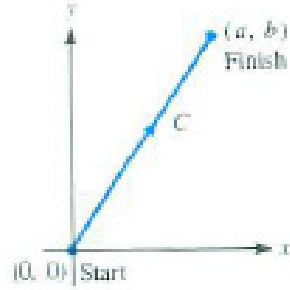


Figure 18.2.3:

Since $f(a, b) = be^a + b^2$, we see that $f(x, y) = ye^x + y^2$ is the desired function. One could check this by showing that the gradient of f is indeed $e^x y\mathbf{i} + (e^x + 2y)\mathbf{j}$. Other suitable potential functions f are $e^x y + y^+ k$ for any constant k.

$ye^x + y^2 + k$ for any constant k, also would be a potential.

◇

The next example uses the *cancellation principle*, which is based on the fact that the sum of two line integrals in opposite direction on a curve is zero. This idea is used here to develop the two-curve version of Green's Theorem and then several more times before the end of this chapter.

Green's Theorem — The Two-Curve Case

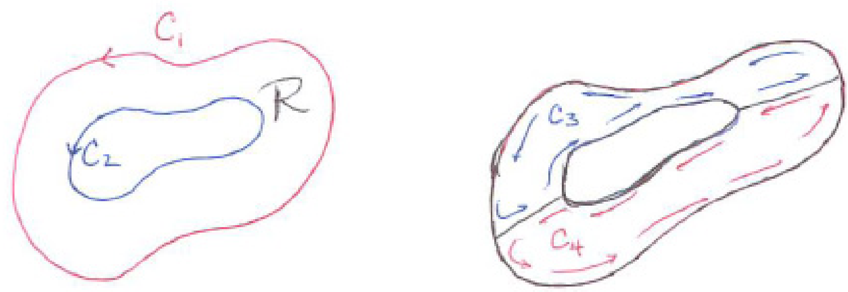
EXAMPLE 3 Figure 18.2.4(a) shows two closed counterclockwise curves C_1, and C_2 that enclose a ring-shaped region $\mathcal{R}$ in which $\nabla \times \mathbf{F}$ is $\mathbf{0}$. Show that the circulation of $\mathbf{F}$ over C_1 equals the circulation of $\mathbf{F}$ over C_2.

SOLUTION Cut $\mathcal{R}$ into two regions, each bounded by a simple curve, to which we can apply Theorem 18.2. Let C_3 bound one of the regions and C_4 bound the other, with the usual counterclockwise orientation. On the cuts, C_3 and C_4 go in opposite directions. On the outer curve C_3 and C_4 have the same orientation as C_1. On the inner curve they are the opposite orientation of C_2. (See Figure 18.1.2(b).) Thus

$$\int_{C_3} \mathbf{F} \cdot\ dr + \int_{C_4} \mathbf{F} \cdot\ dr = \int_{C_1} \mathbf{F} \cdot\ dr - \int_{C_2} \mathbf{F} \cdot\ dr. \tag{18.2.3}$$

By Theorem 18.2 each integral on the left side of (18.2.3) is 0. Thus

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r} \tag{18.2.4}$$



(a) (b)

Figure 18.2.4:

◇

Example 3 justifies the "two-curve" variation of Green's Theorem:

Two-Curve Version of Green's Theorem
Assume two nonoverlapping curves C_1 and C_2 lie in a region where $\mathbf{curl}\,\mathbf{F}$ is $\mathbf{0}$ and form the border of a ring. Then, if C_1 and C_2 both have the same orientation,

$$\oint_{C_1} \mathbf{F} \cdot d\mathbf{r} = \oint_{C_2} \mathbf{F} \cdot d\mathbf{r}.$$

This theorem tells us "as you move a closed curve within a region of zero-curl, you don't change the circulation." The next Example illustrates this point.

EXAMPLE 4 Let $\mathbf{F} = \frac{-y\mathbf{i}+x\mathbf{j}}{x^2+y^2}$ and C be the closed counterclockwise curve bounding the square whose vertices are $(-2,-2)$, $(2,-2)$, $(2,2)$, and $(-2,2)$. Evaluate the circulation of $\mathbf{F}$ around C as easily as possible.
SOLUTION This vector field appeared in Example 5 of Section 18.1. Since its curl is $\mathbf{0}$, at all points except the origin, where $\mathbf{F}$ is not defined, we may use the two-curve version of Green's Theorem. Thus $\oint_C \mathbf{F} \cdot d\mathbf{r}$ equals the circulation of $\mathbf{F}$ over the unit circle in Example 5, hence equals 2π.

This is a lot easier than integrating $\mathbf{F}$ directly over each of the four edges of the square. ◇

How to Draw $\nabla \times \mathbf{F}$

For the planar vector field $\mathbf{F}$, its curl, $\nabla \times \mathbf{F}$, is of the form $z(x,y)\mathbf{k}$. If $z(x,y)$ is positive, the curl points directly up from the page. Indicate this by the symbol $\odot$, which suggests the point of an arrow or the nose of a rocket. If $z(x, y)$ is negative, the curl points down from the page. To show this, use the symbol $\oplus$, which suggests the feathers of an arrow or the fins of a rocket. Figure 18.2.5 illustrates their use.

This is standard notation in physics.

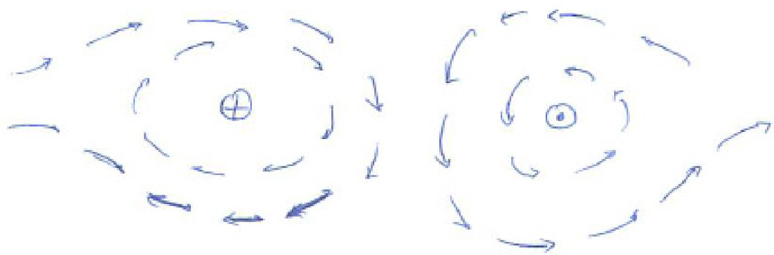


Figure 18.2.5:

Summary

We first expressed Green's theorem in terms of scalar functions

$$\oint_C (P\ dx + Q\ dy) = \int_{\mathcal{R}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\ dA.$$

We then translated it into a statement about the circulation of a vector field;

$$\oint_C \mathbf{F} \cdot\ d\mathbf{r} = \int_{\mathcal{R}} (\nabla \times \mathbf{F}) \cdot \mathbf{k}\ dA.$$

In this theorem the closed curve C is oriented counterclockwise.

With the aid of this theorem we were able to show the following important result:

> If the curl of **F** is **0** *and if the domain of* **F** *is simply connected*, then **F** is conservative.

Also, in a region in which $\nabla \times \mathbf{F} = \mathbf{0}$, the value of $\oint_C \mathbf{F} \cdot\ d\mathbf{r}$ does not change as you gradually change C to other curves in the region.

EXERCISES for Section 18.2 *Key:* R–routine, M–moderate, C–challenging

In Exercises 1 through 4 verify Green's Theorem for the given functions P and Q and curve C.

1.[R] $P = xy$, $Q = y^2$ and C is the border of the square whose vertices are $(0, 0)$, $(1, 0)$, $(1, 1)$ and $(0, 1)$.

2.[R] $P = x^2$, $Q = 0$ and C is the boundary of the unit circle with center $(0, 0)$.

3.[R] $P = e^y$, $Q = e^x$ and C is the triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$.

4.[R] $P = \sin(y)$, $Q = 0$ and C is the boundary of the portion of the unit disk with center $(0, 0)$ in the first quadrant.

5.[R] Figure 18.2.6 shows a vector field for a fluid flow $\mathbf{F}$. At the indicated points A, B, C, and D tell when the curl of $\mathbf{F}$ is pointed up, down or is $\mathbf{0}$. (Use the $\odot$ and $\oplus$ notation.) HINT: When the fingers of your right hand copy the direction of the flow, your thumb points in the direction of the curl, up or down.

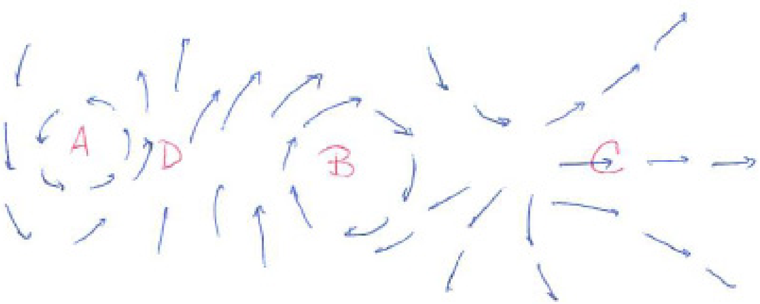


Figure 18.2.6:

6.[R] Assume that $\mathbf{F}$ describes a fluid flow. Let P be a point in the domain of $\mathbf{F}$ and C a small circular path around P.

(a) If the curl of $\mathbf{F}$ points upward, in what direction is the fluid tending to turn near P, clockwise or counterclockwise?

(b) If C is oriented clockwise, would $\oint_C \mathbf{F} \cdot d\mathbf{r}$ to be positive or negative?

7.[R] In Example 2 we constructed a function f by using a straight path from $(0, 0)$ to (a, b). Instead, construct f by using a path that consists of two line segments, the first from $(0, 0)$ to $(a, 0)$, and the second, from $(a, 0)$ to (a, b).

8.[R] In Example 2 we constructed a function f by using a straight path from $(0, 0)$ to (a, b). Instead, construct f by using a path that consists of two line segments, the first from $(0, 0)$ to $(0, b)$, and the second from $(0, b)$ to (a, b).

9.[R] Another way to construct a potential function f for a vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ is to work directly with the requirement that $\nabla f = \mathbf{F}$. That is, with the equattions

$$\frac{\partial f}{\partial x} = P(x, y) \qquad \text{and} \qquad \frac{\partial f}{\partial y} = Q(x, y).$$

(a) Integrate $\frac{\partial f}{\partial x} = e^x y$ with respect to x to conclude that $f(x, y) = e^x y + C(y)$. Note that the "constant of integration" can be any function of y, which we call $C(y)$. (Why?)

(b) Next, differentiate the result found in (a) with respect to y. This gives two formulas for $\frac{\partial f}{\partial y}$: $e^x + C'(y)$ and $e^x + 2y$. Use this fact to explain why $C'(y) = 2y$.

(c) Solve the equation for C found in (b).

(d) Combine the results of (a) and (c) to obtain the general form for a potential function for this vector field.

In Exercises 10 through 13

(a) check that $\mathbf{F}$ is conservative in the given domain, that is $\nabla \times \mathbf{F} = \mathbf{0}$, and the domain of $\mathbf{F}$ is simply connected

(b) construct f such that $\nabla f = \mathbf{F}$, using integrals on curves

(c) construct f such that $\nabla f = \mathbf{F}$, using antiderivatives, as in Exercise 9.

10.[R] $\mathbf{F} = 3x^2y\ vi + x^3\mathbf{j}$, domain the xy-plane

11.[R] $\mathbf{F} = y\cos(xy)\ vi + (x\cos(xy) + 2y)\mathbf{j}$, domain the xy-plane

12.[R] $\mathbf{F} = (ye^{xy} + 1/x)\mathbf{i} + xe^{xy}\mathbf{j}$, domain all xy with $x > 0$

13.[R] $\mathbf{F} = \frac{2y\ln(x)}{x}\mathbf{i} + (\ln(x))^2\mathbf{j}$, domain all xy with $x > 0$

14.[R] Verify Green's Theorem when $\mathbf{F}(xy) = x\mathbf{i} + y\mathbf{j}$ and $\mathcal{R}$ is the disk of radius a and center at the origin.

15.[R] In Example 1 we used Green's Theorem show that $\oint_C y\ dx$ is the negative of the area that encloses. Use Green's Theorem to show that $\oint_C x$ equals that area. (We obtained this result in S tion 15.3 without Green's Theorem.)

16.[R] Let A be a plane region with boundary C simple closed curve swept out counterclockwise. Use Green's theorem to show that the area of A equals

$$\frac{1}{2}\oint(-y\ dx + x\ dy).$$

17.[R] Use Exercise 16 to find the area of the region bounded by the line $y = x$ and the curve

$$\begin{cases} x = t^6 + t^4 \\ y = t^3 + t \end{cases} \quad \text{for } t \text{ in } [0, 1].$$

18.[R] Assume that $\mathbf{curl}\,\mathbf{F}$ at $(0, 0)$ is -3. Let C sweep out the boundary of a circle of radius a, center at $(0, 0)$. When a is small, estimate the circulation $\int_C \mathbf{F} \cdot d\mathbf{r}$.

19.[R] Which of these fields are conservative:

(a) $x\mathbf{i} - y\mathbf{j}$

(b) $\frac{x\mathbf{i} - y\mathbf{j}}{x^2 + y^2}$

(c) $3\mathbf{i} + 4\mathbf{j}$

(d) $(6xy - y^3)\mathbf{i} + (4y + 3x^2 - 3xy^2)\mathbf{j}$

(e) $\frac{y\mathbf{i} - x\mathbf{j}}{1 + x^2y^2}$

(f) $\frac{x\mathbf{i} + y\mathbf{j}}{x^2 + y^2}$

20.[R] Figure 18.2.7 shows a fluid flow $\mathbf{F}$. All the vectors are parallel, but their magnitudes increase from bottom to top. A small simple curve C is placed in the flow.

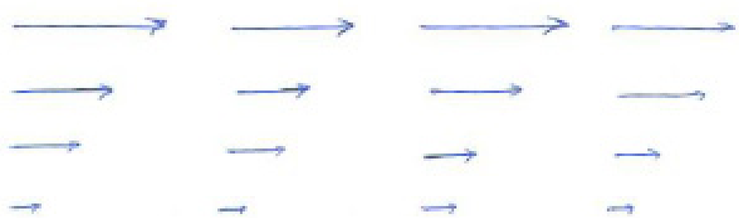


Figure 18.2.7:

(a) Is the circulation around C positive, negative, or 0? Justify your opinion.

(b) Assume that a wheel with small blades is free to rotate around its axis, which is perpendicular to the page. When it is inserted into this flow, which way would it turn, or would it not turn at all? (Don't just say, "It would get wet.")

21.[R] Let $\mathbf{F}(x, y) = y^2\mathbf{i}$.

(a) Sketch the field.

(b) Without computing it, predict when $(\nabla \times \mathbf{F}) \cdot \mathbf{k}$ is positive, negative or zero.

(c) Compute $(\nabla \times \mathbf{F}) \cdot \mathbf{k}$.

(d) What would happen if you dipped a wheel with small blades free to rotate around its axis, which is perpendicular to the page, into this flow.

22.[R] Check that the curl of the vector field in Example 2 is **0**, as asserted.

23.[R] Explain in words, without explicit calculations, why the circulation of the field $f(r)\widehat{\mathbf{r}}$ around the curve $PQRSP$ in Figure 18.2.8 is zero. As usual, f is a scalar function, $r = ||\mathbf{r}||$, and $\widehat{r} = \mathbf{r}/r$.

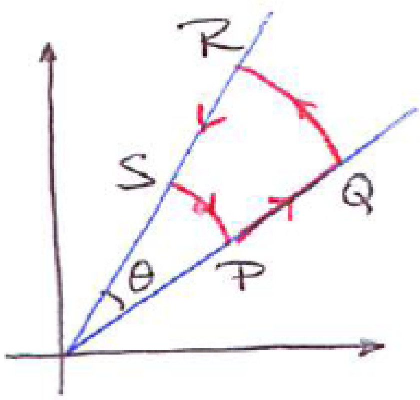


Figure 18.2.8: ARTIST: Please color the four sides of the closed curve.

In Exercises 24 to 27 let **F** be a vector field defined everywhere in the plane except a the point P shown in Figure 18.2.9. Assume that $\nabla \times \mathbf{F} = \mathbf{0}$ and that $\int_{C_1} \mathbf{F} \cdot \, d\mathbf{r} = 5$.

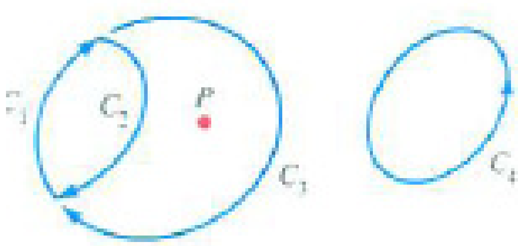


Figure 18.2.9:

24.[R] What, if anything, can be said about $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$?

25.[R] What, if anything, can be said about $\int_{C_3} \mathbf{F} \cdot d\mathbf{r}$?

26.[R] What, if anything, can be said about $\int_{C_4} \mathbf{F} \cdot d\mathbf{r}$?

27.[R] What, if anything, can be said about $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve formed by C_1 followed by C_3?

In Exercises 28 to 31 show that the vector field is conservative and then construct a scalar function of which it is the gradient. Use the method in Example 2.

28.[R] $2xy\mathbf{i} + x^2\mathbf{j}$

29.[R] $\sin(y)\mathbf{i} + (x\cos(y) + 3)\mathbf{j}$

30.[R] $(y+1)\mathbf{i}+(x+1)\mathbf{j}$

31.[R] $3y\sin^2(xy)\cos(xy)\mathbf{i}+(1+3x\sin^2(xy)\cos(xy))\mathbf{j}$

32.[R] Show that

(a) $3x^2y\ dx + x^3\ dy$ is exact.

(b) $3xy\ dx + x^2\ dy$ is not exact.

33.[R] Show that $(x\ dx + y\ dy)/(x^2 + y^2)$ is exact and exhibit a function f such that df equals the given expression. (That is, find f such that $\nabla f \cdot d\mathbf{r}$ agrees with the given differential form.)

34.[R] Let $\mathbf{F} = \widehat{\mathbf{r}}/\|\mathbf{r}\|$ in the xy plane and let C be the circle of radius a and center $(0, 0)$.

(a) Evaluate $\oint_C \mathbf{F} \cdot \mathbf{n}\ ds$ without using Green's theorem.

(b) Let C now be the circle of radius 3 and center $(4, 0)$. Evaluate $\oint_C \mathbf{F} \cdot \mathbf{n}\ ds$, doing as little work as possible.

35.[R] Figure 18.2.10(a) shows the direction of a vector field at three points. Draw a vector field compatible with these values. (No zero-vectors, please.)

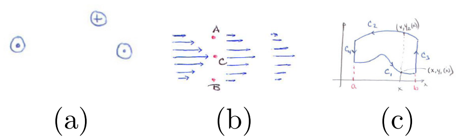


Figure 18.2.10:

36.[R] Consider the vector field in Figure 18.2.10(b). Will a paddle wheel turn at A? At B? At C? If so, in which direction?

37.[R] Use Exercise 16 to obtain the formula for area in polar coordinates:

$$\text{Area} = \frac{1}{2}\int_{\alpha}^{\beta} r^2\, d\theta.$$

Hint: Assume C is given parametrically as $x = r(\theta)\cos(\theta)$, $y = r(\theta)\sin(\theta)$, for $\alpha \le \theta \le \beta$.

38.[M] A curve is given parametrically by $x = t(1-t^2)$, $y = t^2(1-t^3)$, for t in $[0,1]$.

(a) Sketch the points corresponding to $t = 0$, 0.2, 0.4, 0.6, 0.8, and 1.0, and use them to sketch the curve.

(b) Let $\mathcal{R}$ be the region enclosed by the curve. What difficulty arises when you try to compute the area of $\mathcal{R}$ by a definite integral involving vertical or horizontal cross sections?

(c) Use Exercise 16 to find the area of $\mathcal{R}$.

39.[M] Repeat Exercise 38 for $x = \sin(\pi t)$ and $y = t - t^2$, for t in $[0,1]$. In (a), let $t = 0$, $1/4$, $1/2$, $3/4$, and 1.

40.[C] Assume that you know that Green's Theorem is true when $\mathcal{R}$ is a triangle and C its boundary.

(a) Deduce that it therefore holds for quadrilaterals.

(b) Deduce that it holds for polygons.

41.[C] Assume that $\nabla \times \mathbf{F} = \mathbf{0}$ in the region $\mathcal{R}$ bounded by an exterior curve C_1 and two interior curves C_2 and C_3, as in Figure 18.2.11. Show that $\int_{C_1} \mathbf{F}\cdot\, d\mathbf{r} = \int_{C_2} \mathbf{F}\cdot\, d\mathbf{r} + \int_{C_3} \mathbf{F}\cdot\, d\mathbf{r}$.

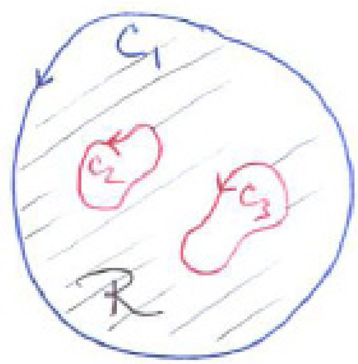


Figure 18.2.11:

42.[C] We proved that $\int_{\mathcal{R}} \frac{\partial Q}{\partial y}\, dA = \int_C Q\, dy$ in a special case. Prove it in this more general case, in which we assume less about the region $\mathcal{R}$. Assume that $\mathcal{R}$ has the description $a \le x \le b$, $y_1(x) \le y \le y_2(x)$. Figure 18.2.10(c) shows such a region, which need not be convex. The curved path C breaks up into four paths, two of which are straight (or may be empty), as in Figure 18.2.10(c).

43.[C] We proved the second part of (18.2.1), namely that $\oint_C Q\, dy = \int_R \partial Q/\partial x\, dA$. Prove the first part, $\oint_C P\, dx = -\int_R \partial p/\partial y\, dA$.

18.3 Green's Theorem, Flux, and Divergence

In the previous section we introduced Green's Theorem and applied it to discover a theorem about circulation and curl. That concerned the line integral of $\mathbf{F} \cdot \mathbf{T}$, the tangential component of $\mathbf{F}$, since $\mathbf{F} \cdot\ d\mathbf{r}$ is short for $(\mathbf{F} \cdot \mathbf{T})\ ds$. Now we will translate Green's Theorem into a theorem about the line integral of $\mathbf{F} \cdot \mathbf{n}$, the normal component of $\mathbf{F}$, $\oint \mathbf{F} \cdot \mathbf{n}\ ds$. Thus Green's Theorem will provide information about the flow of the vector field $\mathbf{F}$ across a closed curve C (see Section 15.4).

Green's Theorem Expressed in Terms of Flux

Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ and C be a counterclockwise closed curve. (We use M and N now, to avoid confusion with P and Q needed later.) At a point on a closed curve the **unit exterior normal vector** (or **unit outward normal vector**) $\mathbf{n}$ is perpendicular to the curve and points outward from the region enclosed by the curve. To compute $\mathbf{F} \cdot \mathbf{n}$ in terms of M and N, we first express $\mathbf{n}$ in terms of $\mathbf{i}$ and $\mathbf{j}$.

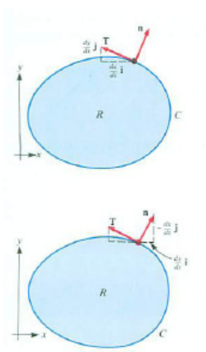


Figure 18.3.1:

The vector

$$\mathbf{T} = \frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j}$$

is tangent to the curve, has length 1, and points in the direction in which the curve is swept out. A typical $\mathbf{T}$ and $\mathbf{n}$ are shown in Figure 18.3.1. As Figure 18.3.1 shows, the exterior unit normal $\mathbf{n}$ has its x component equal to the y component of $\mathbf{T}$ and its y component equal to the negative of the x component of $\mathbf{T}$. Thus

$$\mathbf{n} = \frac{dy}{ds}\mathbf{i} - \frac{dx}{ds}\mathbf{j}.$$

Consequently, if $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$, then

$$\begin{aligned}\oint_C \mathbf{F} \cdot \mathbf{n} &= \oint_C (M\mathbf{i} + N\mathbf{j}) \cdot \left(\frac{dy}{ds}\mathbf{i} - \frac{dx}{ds}\mathbf{j}\right)\ ds = \oint_C \left(M\frac{dy}{ds} - N\frac{dx}{ds}\right)\ ds \\ &= \oint_C (M\ dy - N\ dx) = \oint_C (-N\ dx + M\ dy). \qquad (18.3.1)\end{aligned}$$

In (18.3.1), $-N$ plays the role of P and M plays the role of Q in Green's Theorem. Since Green's Theorem states that

$$\oint_C (P\ dx + Q\ dy) = \int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ dA$$

we have

$$\oint_C (-N\ dx + M\ dy) = \int_R \left(\frac{\partial M}{\partial x} - \frac{\partial(-N)}{\partial y}\right)\ dA$$

or simply, if $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$, then

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right)\, dA.$$

In our customary "P and Q" notations, we have

Green's Theorem Expressed in Terms of Flux

If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$, then

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_{\mathcal{R}} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right)\, dA$$

where C is the boundary of $\mathcal{R}$.

The expression

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y},$$

the sum of two partial derivatives, is call the **divergence** of $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$. It is written $\operatorname{div} \mathbf{F}$ or $\nabla \cdot \mathbf{F}$. The latter notation is suggested by the "symbolic" dot product

$$\left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} \right) \cdot (P\mathbf{i} + Q\mathbf{j}) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}.$$

It is pronounced "del dot eff". Theorem 18.3 is called "the divergence theorem in the plane." It can be written as

Divergence Theorem in the Plane

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_{\mathcal{R}} \operatorname{div} \mathbf{F}\, dA$$

where C is the boundary of $\mathcal{R}$.

EXAMPLE 1 Compute the divergence of (a) $\mathbf{F} = e^{xy}\mathbf{i} + \arctan(3x)\mathbf{j}$ and (b) $\mathbf{F} = -x^2\mathbf{i} + 2xy\mathbf{j}$.

SOLUTION

(a) $\frac{\partial}{\partial x} e^{xy} + \frac{\partial}{\partial y} \arctan(3x) = ye^{xy} + 0 = ye^{xy}$

(b) $\frac{\partial}{\partial x}(-x^2) + \frac{\partial}{\partial y}(2xy) = -2x + 2x = 0.$

$\diamond$

The double integral of the divergence of $\mathbf{F}$ over a region describes the amount of flow across the border of that region. It tells how rapidly the fluid is leaving (diverging) or entering the region (converging). Hence the name "divergence".

In the next section we will be using the divergence of a vector field defined in space, $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$, where P, Q and R are functions of x, y, and z. It is defined as the sum of three partial derivatives

$$\nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

It will play a role in measuring flux across a surface.

EXAMPLE 2 Verify that $\oint_C \mathbf{F} \cdot \mathbf{n}\, ds$ equals $\int_R \nabla \cdot \mathbf{F}\, dA$, when $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}$, R is the disk of radius a and center at the origin and C is the boundary curve of R.

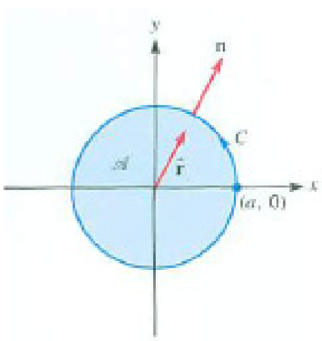


Figure 18.3.2:

SOLUTION First we compute $\oint_C \mathbf{F} \cdot \mathbf{n}\, ds$, where C is the circle bounding $\mathcal{R}$. (See Figure 18.3.2.)

Since C is a circle centered at $(0, 0)$, the unit exterior normal $\mathbf{n}$ is $\widehat{\mathbf{r}}$:

$$\mathbf{n} = \widehat{\mathbf{r}} = \frac{x\mathbf{i} + y\mathbf{j}}{\|x\mathbf{i} + y\mathbf{j}\|} = \frac{x\mathbf{i} + y\mathbf{j}}{a}.$$

Thus, remembering that $\oint_C\, ds$ is just the arclength of C,

$$\begin{aligned} \oint\limits_C \mathbf{F} \cdot \mathbf{n}\, ds &= \oint\limits_C (x\mathbf{i} + y\mathbf{j}) \cdot \left(\frac{x\mathbf{i} + y\mathbf{j}}{a} \right) ds = \oint\limits_C \frac{x^2 + y^2}{a}\, ds \\ &= \oint\limits_C \frac{a^2}{a}\, ds = a \oint\limits_C\, ds = a(2\pi a) = 2\pi a^2. \end{aligned} \tag{18.3.2}$$

Next we compute $\int_{\mathcal{R}} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA$. Since $P = x$ and $Q = y$, $\partial P/\partial x + \partial Q/\partial y = 1 + 1 = 2$. Then

$$\int\limits_{\mathcal{R}} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA = \int\limits_{\mathcal{R}} 2\, dA,$$

which is twice the area of the disk $\mathcal{R}$, hence $2\pi a^2$. This agrees with (18.3.2). $\diamond$

As the next example shows, a double integral can provide a way to compute the flux: $\oint \mathbf{F} \cdot \mathbf{n}\, ds$.

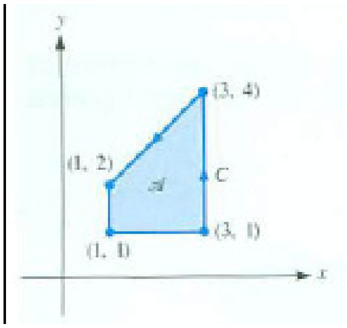


Figure 18.3.3:

EXAMPLE 3 Let $\mathbf{F} = x^2\mathbf{i} + xy\mathbf{j}$. Evaluate $\oint \mathbf{F}\cdot\mathbf{n}\,ds$ over the curve that bounds the quadrilateral with vertices $(1,1)$, $(3,1)$, $(3,4)$, and $(1,2)$ shown in Figure 18.3.3.

SOLUTION The line integral could be evaluated directly, but would require parameterizing each of the four edges of C. With Green's Theorem we can instead evaluate an integral over a single plane region.

Let $\mathcal{R}$ be the region that C bounds. By Green's theorem

$$\begin{aligned}\oint_C \mathbf{F}\cdot\mathbf{n}\,ds &= \int_{\mathcal{R}} \nabla\cdot\mathbf{F}\,dA = \int_{\mathcal{R}} \left(\frac{\partial(x^2)}{\partial x} + \frac{\partial(xy)}{\partial y}\right)\,dA\\ &= \int_{\mathcal{R}} (2x+x)\,dA = \int_{\mathcal{R}} 3x\,dA.\end{aligned}$$

See Exercise 15.

Then

$$\int_{\mathcal{R}} 3x\,dA = \int_1^3\int_1^{y(x)} 3x\,dy\,dx,$$

where $y(x)$ is determined by the equation of the line that provides the top edge of $\mathcal{R}$. We easily find that the line through $(1,2)$ and $(3,4)$ has the equation $y = x+1$. Therefore,

$$\int_{\mathcal{R}} 3x\,dA = \int_1^3\int_1^{x+1} 3x\,dy\,dx.$$

The inner integration gives

$$\int_1^{x+1} 3x\,dy = 3xy\big|_{y=1}^{y=x+1} = 3x(x+1) - 3x = 3x^2.$$

The second integration gives

$$\int_1^3 3x^2\,dx = x^3\big|_1^3 = 27 - 1 = 26$$

◇

A Local View of div F

We have presented a "global" view of $\operatorname{div}\mathbf{F}$, integrating it over a region $\mathcal{R}$ to get the total divergence across the boundary of $\mathcal{R}$. But there is another way of viewing $\operatorname{div}\mathbf{F}$, "locally." This approach makes uses an extension of the Permanence Principle of Section 2.5 to the plane and to space.

Let $P = (a, b)$ be a point in the plane and $\mathbf{F}$ a vector field describing fluid flow. Choose a very small region $\mathcal{R}$ around P, and let C be its boundary. (See Figure 18.3.4.) Then the net flow out of $\mathcal{R}$ is

$$\oint\limits_C \mathbf{F}\cdot\mathbf{n}\, ds.$$

By Green's theorem, the net flow is also

$$\int\limits_{\mathcal{R}} \operatorname{div}\mathbf{F}\, dA.$$

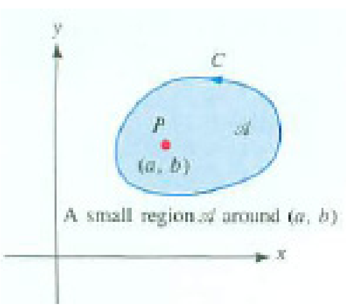


Figure 18.3.4:

Now, since $\operatorname{div}\mathbf{F}$ is continuous and $\mathcal{R}$ is small, $\operatorname{div}\mathbf{F}$ is almost constant throughout $\mathcal{R}$, staying close to the divergence of $\mathbf{F}$ at (a, b). Thus

$$\int\limits_{\mathcal{R}} \operatorname{div}\mathbf{F}\, dA \approx \operatorname{div}\mathbf{F}(a, b)\ \mathrm{Area}(\mathcal{R}).$$

or, equivalently,

$$\frac{\text{Net flow out of } R}{\text{Area of } \mathcal{R}} \approx \operatorname{div}\mathbf{F}(a, b). \tag{18.3.3}$$

This means that

$$\operatorname{div}\mathbf{F} \text{ at } P$$

is a measure of the rate at which fluid tends to leave a small region around P. Hence another reason for the name "divergence." If $\operatorname{div}\mathbf{F}$ is positive, fluid near P tends to get less dense (diverge). If $\operatorname{div}\mathbf{F}$ is negative, fluid near P tends to accumulate (converge).

Moreover, (18.3.3) suggests a different definition of the divergence $\operatorname{div}\mathbf{F}$ at (a, b), namely

Diameter is defined in Section 17.1.

Local Definition of $\operatorname{div}\mathbf{F}(a, b)$

$$\operatorname{div}\mathbf{F}(a, b) = \lim_{\text{Diameter of } \mathcal{R}\to 0} \frac{\oint_C \mathbf{F}\cdot\mathbf{n}\, ds}{\text{Area of } \mathcal{R}}$$

where $\mathcal{R}$ is a region enclosing (a, b) whose boundary C is a simple closed curve.

This definition appeals to our physical intuition. We began by defining $\operatorname{div}\mathbf{F}$ mathematically, as $\partial P/\partial x + \partial Q/\partial y$. We now see its physical meaning, which is independent of any coordinate system. This coordinate-free definition is the basis for Section 18.9.

EXAMPLE 4 Estimate the flux of $\mathbf{F}$ across a small circle C of radius a if $\operatorname{div}\mathbf{F}$ at the center of the circle is 3.

SOLUTION The flux of $\mathbf{F}$ across C is $\oint_C \mathbf{F}\cdot\mathbf{n}\; ds$, which equals $\int_{\mathcal{R}} \operatorname{div}\mathbf{F}\; dA$, where $\mathcal{R}$ is the disk that C bounds. Since $\operatorname{div}\mathbf{F}$ is continuous, it changes little in a small enough disk, and we treat it as almost constant. Then $\int_{\mathcal{R}} \operatorname{div}\mathbf{F}\; dA$ is approximately (3)(Area of $\mathcal{R}$) $= 3(\pi a^2) = 3\pi a^2$. ◇

Proof of Green's Theorem

As Steve Whitaker of the chemical engineering department at the University of California at Davis has observed, "The concepts that one must understand to *prove* a theorem are frequently the concepts one must understand to *apply* the theorem." So read the proof slowly at least twice. It is not here just to show that Green's theorem is true. After all, it has been around for over 150 years, and no one has said it is false. Studying a proof strengthens one's understanding of the fundamentals.

In this proof we use the concepts of a double integral, an iterated integral, a line integral, and the fundamental theorem of calculus. So the proof provides a quick review of four basic ideas.

We prove that $\oint_{\mathcal{R}} Q\; dy = \int_{\mathcal{R}} \frac{\partial Q}{\partial x}\; dA$. The proof that $\oint_C P\; dx = -\int \frac{\partial P}{\partial y}\; dA$ is similar.

To avoid getting involved in distracting details we assume that $\mathcal{R}$ is **strictly convex**: It has no dents and its border has no straight line segments. The basic ideas of the proof show up clearly in this special case. Thus $\mathcal{R}$ has the description $a \le x \le b$, $y_1(x) \le y \le y_2(x)$, as shown in Figure 18.3.5. We will express both $\int_{\mathcal{R}} \frac{\partial Q}{\partial y}\; dA$ and $\int_C Q\; dy$ as definite integrals over the interval $[a,b]$.

First, we have

$$\int\limits_{\mathcal{R}} \frac{\partial Q}{\partial y}\; dA = \int\limits_a^b \int\limits_{y_1(x)}^{y_2(x)} \frac{\partial Q}{\partial y}\; dy\; dx.$$

By the Fundamental Theorem of Calculus,

$$\int\limits_{y_1(x)}^{y_2(x)} \frac{\partial Q}{\partial y}\; dy = Q(x, y_2(x)) - Q(x, y_1(x)).$$

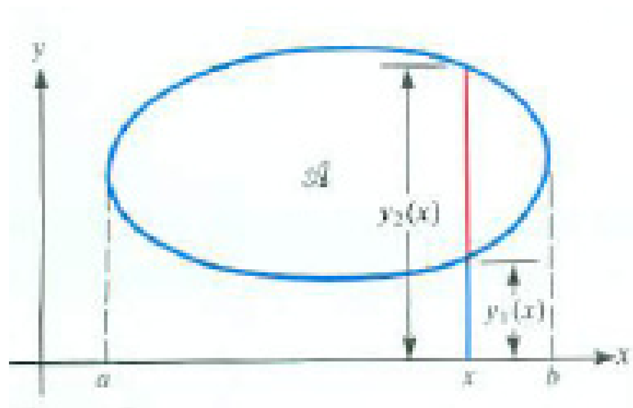


Figure 18.3.5: ARTIST: Please change $\mathcal{A}$ with $\mathcal{R}$.

Hence

$$\int_{\mathcal{R}} \frac{\partial Q}{\partial y}\, dA = \int_a^b \left(Q(x, y_2(x)) - Q(x, y_1(x))\right)\, dx. \tag{18.3.4}$$

Next, to express $\int_C -Q\, dx$ as an integral over $[a.b]$, break the closed path C into two successive paths, one along the bottom part of $\mathcal{R}$, described by $y = y_1(x)$, the other along the top part of $\mathcal{R}$, described by $y = y_2(x)$. Denote the bottom path C_1 and the top path C_2. (See Figure 18.3.6.)

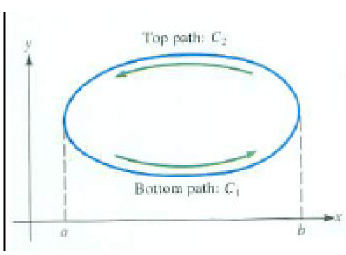


Figure 18.3.6:

Then

$$\oint_C (-Q)\, dx = \int_{C_1} (-Q)\, dx + \int_{C_2} (-Q)\, dx. \tag{18.3.5}$$

But

$$\int_{C_1} (-Q)\, dx = \int_{C_1} \left(-Q(x, y_1(x))\right)\, dx = \int_a^b \left(-Q(x, y_1(x))\right)\, dx,$$

and

$$\int_{C_2} (-Q)\, dx = \int_{C_2} \left(-Q(x, y_2(x))\right)\, dx = \int_b^a \left(-Q(x, y_2(x))\right)\, dx = \int_a^b Q(x, y_2(x))\, dx.$$

Thus by (18.3.5),

$$\begin{aligned}\oint_C (-Q)\, dx &= \int_a^b -Q(x, y_1(x))\, dx + \int_a^b Q(x, y_2(x))\, dx \\ &= \int_a^b \left(Q(x, y_2(x)) - Q(x, y_1(x))\right) dx.\end{aligned}$$

This is also the right side of (18.3.4) and concludes the proof.

Summary

We introduced the "divergence" of a vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$, namely the scalar field $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$ denoted $\operatorname{div} \mathbf{F}$ or $\nabla \cdot \mathbf{F}$.

We translated Green's Theorem into a theorem about the flux of a vector field in the xy-plane. In symbols, the divergence theorem in the plane says that

$$\oint\limits_C \mathbf{F} \cdot \mathbf{n}\, ds = \int\limits_{\mathcal{R}} \operatorname{div} \mathbf{F}\, dA.$$

"The integral of the normal component of $\mathbf{F}$ around a simple closed curve equals the integral of the divergence of $\mathbf{F}$ over the region which the curve bounds."

From this it follows that

$$\operatorname{div} \mathbf{F}(P) = \lim_{\text{diameter of } \mathcal{R} \to 0} \frac{\oint_C \mathbf{F} \cdot \mathbf{n}\, ds}{\text{Area of } \mathcal{R}} = \lim_{\text{diameter of } \mathcal{R} \to 0} \frac{\text{Flux across } C}{\text{Area of } \mathcal{R}}$$

where C is the boundary of the region $\mathcal{R}$, which contains P.

We concluded with a proof of Green's theorem, that provides a review of several basic concepts.

EXERCISES for Section 18.3 *Key:* R–routine, M–moderate, C–challenging

1.[R] State the divergence form of Green's Theorem in symbols.

2.[R] State the divergence form of Green's Theorem in words, using no symbols to denote the vector fields, etc.

In Exercises 3 to 6 compute the divergence of the given vector fields.

3.[R] $\mathbf{F} = x^3y\mathbf{i} + x^2y^3\mathbf{j}$

4.[R] $\mathbf{F} = \arctan(3xy)\mathbf{i} + (e^{y/x})\mathbf{j}$

5.[R] $\mathbf{F} = \ln(x+y)\mathbf{i} + xy(\arcsin y)^2\mathbf{j}$

6.[R] $\mathbf{F} = y\sqrt{1+x^2}\mathbf{i} + \ln((x+1)^3(\sin(y))^{3/5}e^{x+y})\mathbf{j}$

In Exercises 7 to 10 compute $\int_{\mathcal{R}} \operatorname{div}\mathbf{F}\, dA$ and $\oint_C \mathbf{F}\cdot\mathbf{n}\, ds$ and check that they are equal.

7.[R] $\mathbf{F} = 3x\mathbf{i} + 2y\mathbf{j}$, and $\mathcal{R}$ is the disk of radius 1 with center $(0,0)$.

8.[R] $\mathbf{F} = 5y^3\mathbf{i} - 6x^2\mathbf{j}$, and $\mathcal{R}$ is the disk of radius 2 with center $(0,0)$.

9.[R] $\mathbf{F} = xy\mathbf{i} + x^2y\mathbf{j}$, and $\mathcal{R}$ is the square with vertices $(0,0)$, $(a,0)$ (a,b) and $(0,b)$, where a, $b > 0$.

10.[R] $\mathbf{F} = \cos(x+y)\mathbf{i} + \sin(x+y)\mathbf{j}$, and $\mathcal{R}$ is the triangle with vertices $(0,0)$, $(a,0)$ and (a,b), where a, $b > 0$.

In Exercises 11 to 14 use Green's Theorem expressed in terms of divergence to evaluate $\oint_C \mathbf{F}\cdot\mathbf{n}\, ds$ for the given $\mathbf{F}$, where C is the boundary of the given region R.

11.[R] $\mathbf{F} = e^x\sin y\mathbf{i} + e^{2x}\cos(y)\mathbf{j}$, and R is the rectangle with vertices $(0,0)$, $(1,0)$, $(0,\pi/2)$, and $(1,\pi/2)$.

12.[R] $\mathbf{F} = y\tan(x)\mathbf{i} + y^2\mathbf{j}$, and R is the square with vertices $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$.

13.[R] $\mathbf{F} = 2x^3y\mathbf{i} - 3x^2y^2\mathbf{j}$, and R is the triangle with vertices $(0,1)$, $(3,4)$, and $(2,7)$.

14.[R] $\mathbf{F} = \frac{-\mathbf{i}}{xy^2} + \frac{\mathbf{j}}{x^2y}$, and R is the triangle with vertices $(1,1)$, $(2,2)$, and $(1,2)$. Hint: Write $\mathbf{F}$ with a common denominator.

15.[R] In Example 3 we found $\oint_C \mathbf{F}\cdot\mathbf{n}\, ds$ by computing a double integral. Instead, evaluate the integral $\oint_C \mathbf{F}\cdot\mathbf{n}\, ds$ directly.

16.[R] Let $\mathbf{F}(x,y) = \mathbf{i}$, a constant field.

(a) Evaluate directly the flux of $\mathbf{F}$ around the triangular path, $(0,0)$ to $(1,0)$, to $(0,1)$ back to $(0,0)$.

(b) Use the divergence of $\mathbf{F}$ to evaluate the flux in (a).

17.[R] Let a be a "small number" and $\mathcal{R}$ be the square with vertices (a,a), $(-a,a)$, $(-a,-a)$, and $(a,-a)$, and C its boundary. If the divergence of $\mathbf{F}$ at the origin is 3, estimate $\oint_C \mathbf{F}\cdot\mathbf{n}\, ds$.

18.[R] Assume $\|\mathbf{F}(P)\| \leq 4$ for all points P on a curve of length L that bounds a region $\mathcal{R}$ of area A. What can be said about the integral $\int_{\mathcal{R}} \nabla\cdot\mathbf{F}\, dA$?

19.[R] Verify the divergence form of Green's Theorem for $\mathbf{F} = 3x\mathbf{i} + 4y\mathbf{j}$ and C the square whose vertices are $(2,0)$, $(5,0)$, $(5,3)$, and $(2,3)$.

A vector field $\mathbf{F}$ is said to be **divergence free** when $\nabla\cdot\mathbf{F} = 0$ at every point in the field.

20.[R] Figure 18.3.7 shows four vector fields. Two are divergence-free and two are not. Decide which two are not, copy them onto a sheet of drawing paper, and sketch a closed curve C for which $\oint_C \mathbf{F}\cdot\mathbf{n}\, ds$ is not 0.

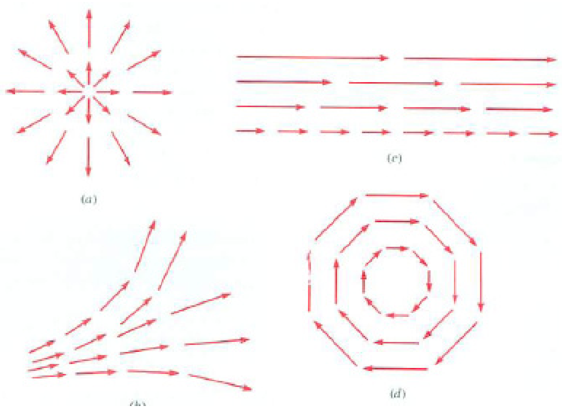


Figure 18.3.7:

21.[R] For a vector field **F**,

(a) Is the curl of the gradient of **F** always **0**?

(b) Is the divergence of the gradient of **F** always 0?

(c) Is the divergence of the curl of **F** always 0?

(d) Is the gradient of the divergence of **F** always **0**?

22.[R] Figure 18.3.8 describes the flow **F** of a fluid. Decide whether $\nabla \cdot \mathbf{F}$ is positive, negative, or zero at each of the points A, B, and C.

Figure 18.3.8:

23.[R] If $\operatorname{div} \mathbf{F}$ at $(0.1, 0.1)$ is 3 estimate $\oint_C \mathbf{F} \cdot \mathbf{n}\, ds$, where C is the curve around the square whose vertices are $(0, 0)$, $(0.2, 0)$, $(0.2, 0.2)$, $(0, 0.2)$.

24.[M] Find the area of the region bounded by the line $y = x$ and the curve

$$\begin{cases} x = t^6 + t^4 \\ y = t^3 + t \end{cases}$$

for t in $[0, 1]$. HINT: Use Green's Theorem.

25.[M] Let f be a scalar function. Let $\mathcal{R}$ be a convex region and C its boundary taken counterclockwise. Show that

$$\int_{\mathcal{R}} \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) dA = \oint_C \left(\frac{\partial f}{\partial x}\, dy - \frac{\partial f}{\partial x}\, dx \right).$$

26.[M] Let **F** be the vector field whose formula in polar coordinates is $\mathbf{F}(r, \theta) = r^n \widehat{\mathbf{r}}$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, $r = \|\mathbf{r}\|$, and $\widehat{\mathbf{r}} = \mathbf{r}/r$. Show that the divergence of **F** is $(n + 1)r^{n-1}$. HINT: First express **F** in rectangular coordinates. NOTE: See also Exercise 46 in Section 18.8.

27.[M] A region with a hole is bounded by two oriented curves C_1 and C_2, as in Figure 18.3.9. which shows typical exterior-pointing unit normal vectors.

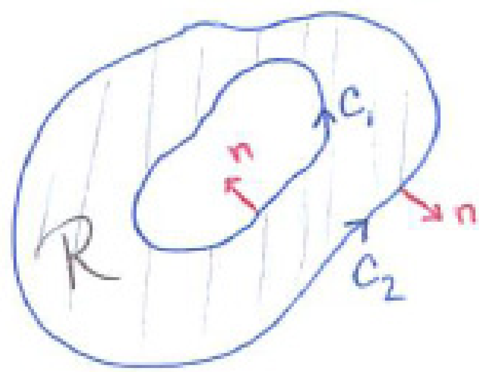


Figure 18.3.9:

Find an equation expressing $\int_{\mathcal{R}} \nabla \cdot \mathbf{F}\, dA$ in terms of $\oint_{C_1} \mathbf{F} \cdot \mathbf{n}\, ds$ and $\oint_{C_2} \mathbf{F} \cdot \mathbf{n}\, ds$. HINT: Break R into two regions that have no holes, as in Exercises 34 and 35.

28.[M] The region R is bounded by the curves C_1 and C_2, as in Figure 18.3.10.

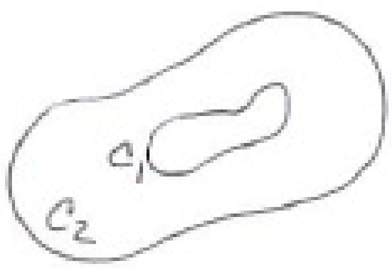
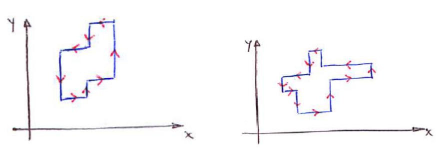


Figure 18.3.10:

(a) Show that $\oint_{C_1} \mathbf{F}\cdot\mathbf{n}\,ds - \int_{C_2} \mathbf{F}\cdot\mathbf{n}\,ds = \int_R (\nabla\cdot\mathbf{F})\,dA$.

(b) If $\nabla\cdot\mathbf{F} = 0$ in $\mathcal{R}$, show that $\int_{C_1} \mathbf{F}\cdot\mathbf{n}\,ds = \int_{C_2} \mathbf{F}\cdot\mathbf{n}\,ds$.

29.[M] Let $\mathbf{F}$ be a vector field in the xy-plane whose flux across any rectangle is 0. Show that its flux across the curves in Figure 18.3.11(a) and (b) is also 0.



(a) (b)

Figure 18.3.11:

30.[M] Assume that the circulation of $\mathbf{F}$ along every circle in the xy-plane is 0. Must $\mathbf{F}$ be conservative?

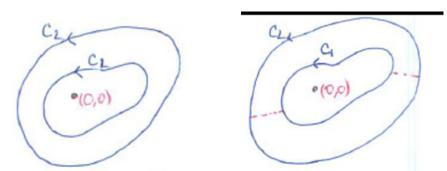
31.[C] The field $\mathbf{F}$ is defined throughout the xy-plane. If the flux of $\mathbf{F}$ across every circle is 0, must the flux of $\mathbf{F}$ across every square be 0? Explain.

32.[C] Let $\mathbf{F}(x,y)$ describe a fluid flow. Assume $\nabla\cdot\mathbf{F}$ is never 0 in a certain region R. Show that none of the stream lines in the region closes up to form a loop within $\mathcal{R}$. HINT: At each point P on a stream line, $\mathbf{F}(P)$ is tangent to that streamline.

33.[C] Let $\mathcal{R}$ be a region in the xy-plane bounded by the closed curve C. Let $f(x,y)$ be defined on the plane. Show that

$$\int_{\mathcal{R}} \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x^2}\right)\,dA = \oint_C D_{\mathbf{n}}(f)\,ds.$$

34.[C] Assume that $\mathbf{F}$ is defined everywhere in the xy-plane except at the origin and that the divergence of $\mathbf{F}$ is identically 0. Let C_1 and C_2 be two counterclockwise simple curves circling the origin. C_1 lies within the region within C_2. Show that $\oint_{C_1} \mathbf{F}\cdot\mathbf{n}\,ds = \int_{C_2} \mathbf{F}\cdot\mathbf{n}\,ds$. (See Figure 18.3.12(a).)



(a) (b)

Figure 18.3.12:

HINT: Draw the dashed lines in Figure 18.3.12(b) to cut the region between C_1 and C_2 into two regions.

35.[C] (This continues Exercise 34.) Assume that $\mathbf{F}$ is defined everywhere in the xy-plane except at the origin and that the divergence of $\mathbf{F}$ is identically 0. Let C_1 and C_2 be two counterclockwise simple curves circling the origin. They may intersect. Show that $\oint_{C_1} \mathbf{F}\cdot\mathbf{n}\,ds = \oint_{C_2} \mathbf{F}\cdot\mathbf{n}\,ds$. The message from this Exercise is this: if the divergence of $\mathbf{F}$ is 0, you are permitted to replace an integral over a complicated curve by an integral over a simpler curve.

36.[C]

(a) Draw enough vectors for the field $\mathbf{F}(x,y) = (x\mathbf{i} + y\mathbf{j})/(x^2+y^2)$ to show what it looks like.

(b) Compute $\nabla\cdot\mathbf{F}$.

(c) Does your sketch in (a) agree with what you found for $\nabla \cdot \mathbf{F}$. in (b)? (If not, redraw the vector field.)

18.4 Central Fields and Steradians

Central fields are a special but important type of vector field that appear in the study of gravity and the attraction or repulsion of electric charges. These fields radiate from a point mass or point charge. Physicists invented these fields in order to avoid the mystery of "action at a distance." One particle acts on another directly, through the vector field it creates. This comforts students of gravitation and electromagnetism by glossing over the riddle of how an object can act upon another without any intervening object such as a rope or spring.

Figure 18.4.1:

Central Fields

A **central field** is a continuous vector field defined everywhere in the plane (or in space) except, perhaps, at a point $\mathcal{O}$, with these two properties:

1. Each vector points towards (or away from) $\mathcal{O}$.

2. The magnitudes of all vectors at a given distance from $\mathcal{O}$ are equal.

$\mathcal{O}$ is call the center, or pole, of the field. A central field is also called "radially symmetric." There are various ways to think of a central vector field. For such a field in the plane, all the vectors at points on a circle with center $\mathcal{O}$ are perpendicular to the circle and have the same length, as shown in Figures 18.4.1 and 18.4.2.

The same holds for central vector fields in space, with "circle" replaced by "sphere."

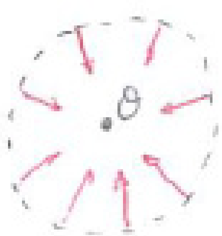


Figure 18.4.2:

The formula for a central vector field has a particularly simple form. Let the field be $\mathbf{F}$ and P any point other than $\mathcal{O}$. Denote the vector $\overrightarrow{OP}$ by $\mathbf{r}$ and its magnitude by r and $\mathbf{r}/r$ by $\widehat{\mathbf{r}}$. Then there is a scalar function f, defined for all positive numbers, such that

$$\mathbf{F}(P) = f(r)\widehat{\mathbf{r}}.$$

The magnitude of $\mathbf{F}(P)$ is $\|f(r)\|$. If $f(r)$ is positive, $\mathbf{F}(P)$ points away from $\mathcal{O}$. If $f(r)$ is negative, $\mathbf{F}(P)$ points toward $\mathcal{O}$.

To conclude this introduction to central fields we point out that a central field is a vector-valued function of more than one variable. Because the point P with coordinates (x, y, z) is also associated with the vector $\mathbf{r} = \overrightarrow{OP} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ we may denote $\mathbf{F}(P)$ as $\mathbf{F}(x, y, z)$ or $\mathbf{F}(\mathbf{r})$.

Central Vector Fields in the Plane

Using polar coordinates with pole placed at the point $\mathcal{O}$, we may express a central field in the form

$$\mathbf{F}(\mathbf{r}) = f(r)\widehat{\mathbf{r}},$$

where $r = \|\mathbf{r}\|$ and $\widehat{\mathbf{r}} = \mathbf{r}/r$. The magnitude of $\mathbf{F}(\mathbf{r})$ is $|f(r)|$.

We already met such a field in Section 18.1 in the study of line integrals. In that case, $f(r) = 1/r$; the "field varied as the inverse first power." When, in Section 15.4, we encountered the line integral for the normal component of this field along a curve we found that it gives the number of radians the curve subtends.

See page 1051.

The vector field $\mathbf{F}(\mathbf{r}) = (1/r)\widehat{\mathbf{r}}$ can also be written as

$$\mathbf{F}(\mathbf{r}) = \frac{\mathbf{r}}{r^2}. \tag{18.4.1}$$

When glancing too quickly at (18.4.1), you might think its magnitude is inversely proportional to the square of r. However, the magnitude of the vector $\mathbf{r}$ in the numerator is r; the magnitude of $\mathbf{r}/r^2$ is $r/r^2 = 1/r$, the reciprocal of the first power of r.

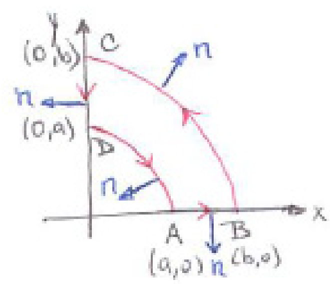


Figure 18.4.3:

EXAMPLE 1 Evaluate the flux $\oint_C \mathbf{F}\cdot\mathbf{n}\,ds$ for the central field $\mathbf{F}(x,y) = f(r)\widehat{\mathbf{r}}$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, over the closed curve shown in Figure 18.4.3. We have $a < b$ and the path goes from $A = (a,0)$ to $B = (b,0)$ to $C = (0,b)$, to $D = (0,a)$ and ends at $A = (a,0)$.

SOLUTION On the paths from A to B and from C to D the exterior normal, $\mathbf{n}$, is perpendicular to $\mathbf{F}$, so $\mathbf{F}\cdot\mathbf{n} = 0$, and these integrands contribute nothing to the integral. On BC, $\mathbf{F}$ equals $f(b)\widehat{\mathbf{r}}$. There $\widehat{\mathbf{r}} = \mathbf{n}$, so $\mathbf{F}\cdot\mathbf{n} = f(b)$ since $\mathbf{r}\cdot\mathbf{n} = 1$. Note that the length of arc BC is $(2\pi b)/4 = \pi b/2$. Thus

$$\int_B^C \mathbf{F}\cdot\mathbf{n}\,ds = \int_B^C f(b)\,ds = f(b)\int_B^C ds == \frac{\pi b}{2} f(b)$$

On the arc DC, $\widehat{\mathbf{r}} = -\mathbf{n}$. A similar calculation shows that

$$\int_D^C \mathbf{F}\cdot\mathbf{n}\,ds = -\frac{\pi}{2} a f(a).$$

Hence

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds \;=\; 0 + \frac{\pi}{2} b f(b) + 0 - \frac{\pi}{2} a f(a) = \frac{\pi}{2}(b f(b) - a f(a)).$$

◇

In order for a central field $f(r)\widehat{\mathbf{r}}$ to have zero flux around all paths of the special type shown in Figure 18.4.3, we must have

$$f(b)b - f(a)a = 0,$$

for all positive a and b. In particular,

$$f(b)b - f(1)1 = 0 \qquad \text{or} \qquad f(b) = \frac{f(1)}{b}.$$

Thus $f(r)$ must be inversely proportional to r and there is a constant c such that

$$f(r) = \frac{c}{r}.$$

If $f(r)$ is not of the form c/r, the vector field $\mathbf{F}(x,y) = f(r)\widehat{\mathbf{r}}$ does not have zero flux across these paths. In Exercise 5 you may compute the divergence of $(c/r)\widehat{\mathbf{r}}$ and show that it is zero.

The only central vector fields with center at the origin in the plane with zero divergence are these whose magnitude is inversely proportional to the distance from the origin.

We underline "in the plane," because in space the only central fields with zero flux across closed surfaces have a magnitude inversely proportional to the square of the distance to the pole, as we will see in a moment.

Knowing that the central field $\mathbf{F} = \widehat{\mathbf{r}}/r$ has zero divergence enables us to evaluate easily line integrals of the form $\oint_C \frac{\widehat{\mathbf{r}}\cdot\mathbf{n}}{r}\,ds$, as the next example shows.

EXAMPLE 2 Let $\mathbf{F}(\mathbf{r}) = \widehat{\mathbf{r}}/r$. Evaluate $\oint_C \mathbf{F}\cdot\mathbf{n}\,ds$ where C is the counterclockwise circle of radius 1 and center $(2,0)$, as shown in Figure 18.4.4.

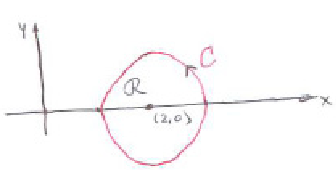


Figure 18.4.4:

SOLUTION Exercise 5 shows that the field $\mathbf{F}$ has 0-divergence throughout C and the region R that C bounds. By Green's Theorem, the integral also equals the integral of the divergence over R:

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_R \nabla\cdot\mathbf{F}\,dA. \tag{18.4.2}$$

Since the divergence of $\mathbf{F}$ is 0 throughout R, the integral on the right side of (18.4.2) is 0. Therefore $\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = 0$. ◇

The next example involves a curve that surrounds a point where the vector field $\mathbf{F} = \widehat{\mathbf{r}}/r$ is not defined.

EXAMPLE 3 Let C be a simple closed curve enclosing the origin. Evaluate $\oint_C \mathbf{F}\cdot\mathbf{n}\, ds$, where $\mathbf{F} = \widehat{\mathbf{r}}/r$.

SOLUTION Figure **??** shows C and a small circle D centered at the origin and situated in the region that C bounds. Without a formula describing C, we could not compute $\oint_C \mathbf{F}\cdot\mathbf{n}\, ds$ directly. However, since the divergence of $\mathbf{F}$ is 0 throughout the region bounded by C and D, we have, by the Two-Curve Case of Green's Theorem,

See page 1051 in Section 15.4.

$$\oint\limits_C \mathbf{F}\cdot\mathbf{n}\, ds = \oint\limits_D \mathbf{F}\cdot\mathbf{n}\, ds. \tag{18.4.3}$$

The integral on the right-hand side of (18.4.3) is easy to compute directly. To do so, let the radius of D be a. Then for points P on D, $\mathbf{F}(P) = \widehat{\mathbf{r}}/a$. Now, $\widehat{\mathbf{r}}$ and $\mathbf{n}$ are the same unit vector. So $\widehat{\mathbf{r}}\cdot\mathbf{n} = 1$. Thus

$$\oint\limits_D \mathbf{F}\cdot\mathbf{n}\, ds = \oint\limits_D \frac{\widehat{\mathbf{r}}\cdot\mathbf{n}}{a}\, ds = \int\limits_D \frac{1}{a}\, ds = \frac{1}{a}2\pi a = 2\pi.$$

Hence $\oint_C \mathbf{F}\cdot\mathbf{n}\, ds = 2\pi$. ◇

Central Fields in Space

A central field in space with center at the origin has the form $\mathbf{F}(x, y, z) = \mathbf{F}(\mathbf{r}) = f(r)\widehat{\mathbf{r}}$ We show that if the flux of $\mathbf{F}$ over any surface bounding certain special regions is zero then $f(r)$ must be inversely proportional to the square of r.

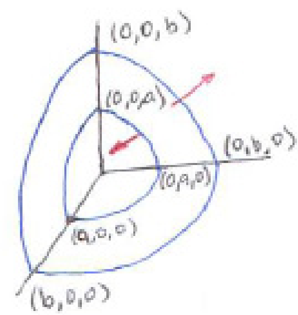


Figure 18.4.5:

Consider the surface S shown in Figure 18.4.5. It consists of an octant of two concentric spheres, one of radius a, the other of radius b, $a < b$, together with the flat surfaces on the coordinate planes. Let $\mathcal{R}$ be the region bounded by the surface S. On its three flat sides $\mathbf{F}$ is perpendicular to the exterior normal. On the outer sphere $\mathbf{F}(x, y, z)\cdot\mathbf{n} = f(b)$. On the inner sphere $\mathbf{F}(x, y, z)\cdot\mathbf{n} = -f(a)$. Thus

Surface area of a sphere of radius r is $4\pi r^2$.

$$\oint\limits_{\mathcal{S}} \mathbf{F}\cdot\mathbf{n}\, dS = f(b)(\frac{1}{8})(4\pi b^2) - f(a)(\frac{1}{8})(4\pi a^2) = \frac{\pi}{2}(f(b)b^2 - f(a)a^2).$$

Since this is to be 0 for all positive a and b, it follows that there is a constant c, such that

Compare with Example 1.

$$f(r) = \frac{c}{r^2}.$$

The magnitude must be proportional to the "inverse square."

The following fact is justified in Exercise 28:

The only central vector field with center at the origin in the plane with zero divergence are these whose magnitude is inversely proportional to the distance from the origin.

A Geometric Application

See Sections 18.7 and 18.9.

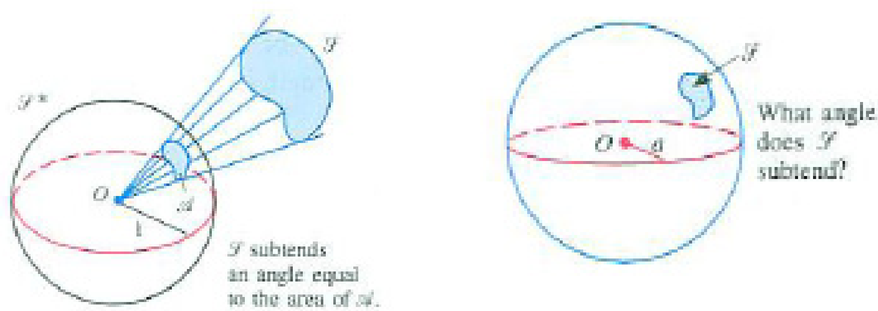
As we will see later in this chapter an "inverse square" central field is at the heart of gravitational theory and electrostatics. Now we show how it is used in geometry, a result we will apply in both areas.

In Section 15.4 we showed how radian measure could be expressed in terms of the line integral $\int_C (\widehat{\mathbf{r}}/r) \cdot \mathbf{n}\, ds$, that is, in terms of the central field whose magnitude is inversely proportional to the *first power* of the distance from the center. That was based on circular arcs in a plane. Now we move up one dimension and consider patches on surfaces of spheres, which will help us measure solid angles.

Let $\mathcal{O}$ be a point and $\mathcal{S}$ a surface such that each ray from $\mathcal{O}$ meets $\mathcal{S}$ in at most one point. Let $\mathcal{S}^*$ be the unit sphere with center at $\mathcal{O}$. The rays from $\mathcal{O}$ that meet $\mathcal{S}$ intersect $\mathcal{S}^*$ in a set that we call $\mathcal{R}$, as shown in Figure 18.4.6(a). Let the area of $\mathcal{R}$ be A. The solid angle subtended by $\mathcal{S}$ at $\mathcal{O}$ is said to have a measure of A steradians

Steradians comes from *stereo*, the Greek word for space, and *radians*.

For instance, a closed surface $\mathcal{S}$ that encloses $\mathcal{O}$ subtends a solid angle of 4π steradians, because the area of the unit sphere is 4π.



(a) (b)

Figure 18.4.6:

EXAMPLE 4 Let $\mathcal{S}$ be part of the surface of a sphere of radius a, $\mathcal{S}_a$, whose center is $\mathcal{O}$. Find the angle subtended by $\mathcal{S}$ at $\mathcal{O}$. (See Figure 18.4.6(b).)

SOLUTION The entire sphere $\mathcal{S}_a$ subtends an angle of 4π steradians because it has an area $4\pi a^2$. We therefore have the proportion

$$\frac{\text{Angle } \mathcal{S} \text{ subtends}}{\text{Angle } \mathcal{S}_a \text{ subtends}} = \frac{\text{Area of } \mathcal{S}}{\text{Area of } \mathcal{S}_a},$$

or

$$\frac{\text{Angle } \mathcal{S} \text{ subtends}}{4\pi} = \frac{\text{Area of } \mathcal{S}}{4\pi a^2}.$$

Hence

$$\text{Angle } \mathcal{S} \text{ subtends} = \frac{\text{Area of } \mathcal{S}}{a^2} \quad \text{steradians.}$$

◇

EXAMPLE 5 Let $\mathcal{S}$ be a surface such that each ray from the point $\mathcal{O}$ meets $\mathcal{S}$ in at most one point. Find an integral that represents in steradians the solid angle that $\mathcal{S}$ subtends at $\mathcal{O}$.

SOLUTION Consider a very small patch of $\mathcal{S}$. Call it $d\mathcal{S}$ and let its area be dA. If we can estimate the angle that this patch subtends at $\mathcal{O}$, then we will have the local approximation that will tell us what integral represents the total solid angle subtended by $\mathcal{S}$.

Let $\mathbf{n}$ be a unit normal at a point in the patch, which we regard as essentially flat, as in Figure 18.4.7. Let $d\mathcal{A}$ be the projection of the patch $d\mathcal{S}$ on a plane perpendicular to $\mathbf{r}$, as shown in Figure 18.4.7. The area of $d\mathcal{A}$ is approximately dA, where

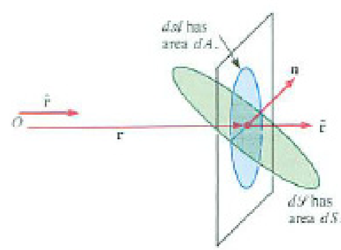


Figure 18.4.7:

$$dA = \widehat{\mathbf{r}} \cdot \mathbf{n}\, dS.$$

Now, $d\mathcal{S}$ and $d\mathcal{A}$ subtend approximately the same solid angle, which according to Example 4 is about

$$\frac{\widehat{\mathbf{r}} \cdot \mathbf{n}}{\|\mathbf{r}\|^2}\, dS \quad \text{steradians.}$$

Consequently $\mathcal{S}$ subtends a solid angle of

$$\int\limits_{\mathcal{S}} \frac{\widehat{\mathbf{r}} \cdot \mathbf{n}}{\|\mathbf{r}\|^2}\, dS \quad \text{steradians.}$$

◇

The following special case will be used in Section 18.5.

Let $\mathcal{O}$ be a point in the region bounded by the closed surface $\mathcal{S}$. Assume each ray from $\mathcal{O}$ meets $\mathcal{S}$ in exactly one point, and let $\mathbf{r}$ denote the position vector from $\mathcal{O}$ to that point. Then

$$\int\limits_{\mathcal{S}} \frac{\widehat{\mathbf{r}} \cdot \mathbf{n}}{r^2}\, dS = 4\pi. \tag{18.4.4}$$

Incidentally, (18.4.4) is easy to establish when $\mathcal{S}$ is a sphere of radius a and center at the origin. In that case $\widehat{\mathbf{r}} = \mathbf{n}$, so $\widehat{\mathbf{r}} \cdot \mathbf{n} = 1$. Also, $r = a$. Then (18.4.4) becomes $\int_{\mathcal{S}} (1/a^2)\, dS = (1/a^2)4\pi a^2 = 4\pi$. However, it is not obvious that (18.4.4) holds far more generally, for instance when $\mathcal{S}$ is a sphere and the origin is *not* its center, or when $\mathcal{S}$ is not a sphere.

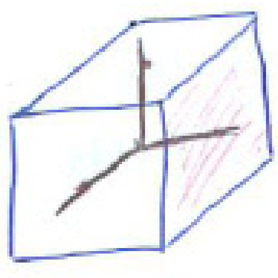


Figure 18.4.8:

EXAMPLE 6 Let $\mathcal{S}$ be the cube of side 2 bounded by the six planes $x = \pm 1$, $y = \pm 1$, $z = \pm 1$, shown in Figure 18.4.8. Find $\oint_{\mathcal{S}} \frac{\widehat{\mathbf{r}} \cdot \mathbf{n}}{r^2}\, dS$, where $\mathcal{S}$ is one of the six faces of the cube.

SOLUTION Each of the six faces subtends the same solid angle at the origin. Since the entire surface subtends 4π steradians, each face subtends $4\pi/6 = 2\pi/3$ steradians. Then the flux over each face is

$$\int\limits_{\mathcal{S}} \frac{\widehat{\mathbf{r}} \cdot \mathbf{n}}{r^2}\, dS = \frac{2\pi}{3}.$$

$\diamond$

Figure 18.4.9:

In physics books you will see the integral $\int_{\mathcal{S}} \frac{\widehat{\mathbf{r}} \cdot \mathbf{n}}{r^2}\, dS$ written using other notations, including:

$$\int\limits_{\mathcal{S}} \frac{\widehat{\mathbf{r}} \cdot \mathbf{n}}{r^3}\, dS, \qquad \int\limits_{\mathcal{S}} \frac{\widehat{\mathbf{r}} \cdot d\mathbf{S}}{r^2}, \qquad \int\limits_{\mathcal{S}} \frac{\mathbf{r} \cdot d\mathbf{S}}{r^3}, \qquad \int\limits_{\mathcal{S}} \frac{\cos(\mathbf{r}, \mathbf{n})}{r^2}\, dS.$$

The symbol $d\mathbf{S}$ is short for $\mathbf{n}\, dS$, and calls to mind Figure 18.4.9, which shows a small patch on the surface, together with an exterior normal unit vector.

Recall that $\cos(\mathbf{r}, \mathbf{n})$ denotes the cosine of the angle between $\mathbf{r}$ and $\mathbf{n}$; see also Section 14.2.

Summary

We investigated central vector fields. In the plane the only divergence-free central fields are of the form $(c/r)\widehat{\mathbf{r}}$ where c is a constant, "an inverse first power." In space the only incompressible central fields are of the form $(c/r^2)\widehat{\mathbf{r}}$, "an inverse second power." The field $\widehat{\mathbf{r}}/r^2$ can be used to express the size of a solid angle of a surface $\mathcal{S}$ in steradians as an integral: $\int_{\mathcal{S}} \widehat{\mathbf{r}} \cdot \mathbf{n}/r^2 \; dS$. In particular, if $\mathcal{S}$ encloses the center of the field, then $\int_{\mathcal{S}} \widehat{\mathbf{r}} \cdot \mathbf{n}/r^2 \; dS = 4\pi$.

Incompressible vector fields have divergence zero, and are discussed again in Section 18.6.

EXERCISES for Section 18.4 *Key:* R–routine, M–moderate, C–challenging

1.[R] Define a central field in words, using no symbols.

2.[R] Define a central field with center at $\mathcal{O}$, in symbols.

3.[R] Give an example of a central field in the plane that

(a) does not have zero divergence,

(b) that does have zero divergence.

4.[R] Give an example of a central field in space that

(a) that is not divergence-free,

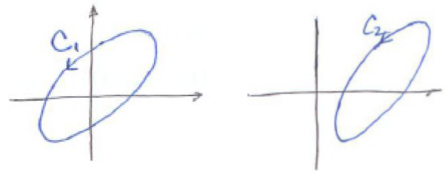
(b) that is divergence-free.

5.[R] Let $\mathbf{F}(x,y)$ be an inverse-first-power central field in the plane $\mathbf{F}(x,y) = (c/r)\widehat{\mathbf{r}}$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$. Compute the divergence of $\mathbf{F}$. Hint: First write $\mathbf{F}(x,y)$ as $\frac{cx\mathbf{i}+cy\mathbf{j}}{x^2+y^2}$.

6.[R] Show that the curl of a central vector field in the plane is $\mathbf{0}$.

7.[R] Show that the curl of a central vector field in space is $\mathbf{0}$.

8.[R] Let $\mathbf{F}(\mathbf{r}) = \widehat{\mathbf{r}}/r$. Evaluate $\oint_C \mathbf{F}\cdot\mathbf{n}\ ds$ as simply as you can for the two ellipses in Figure 18.4.10.



(a) (b)

Figure 18.4.10:

9.[R] Figure 18.4.11 shows a cube of side 2 with one corner at the origin.

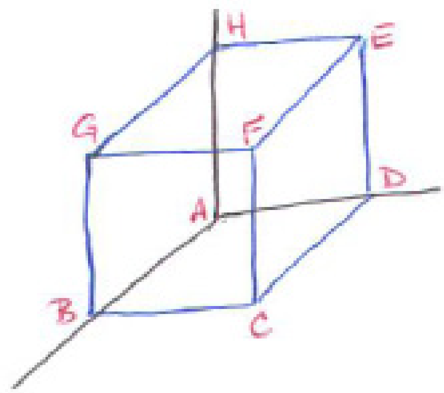


Figure 18.4.11:

Evaluate as easily as you can the integral of the function $\widehat{\mathbf{r}}\cdot\mathbf{n}/r^2$ over

(a) the square $EFGH$,

(b) the square $ABCD$,

(c) the entire surface of the cube.

10.[R] Let $\mathbf{F}(\mathbf{r}) = \widehat{\mathbf{r}}/r^3$. Evaluate the flux of $\mathbf{F}$ over the sphere of radius 2 and center at the origin.

11.[R] A pyramid is made of four congruent equilateral triangles. Find the steradians subtended by one face at the centroid of the pyramid. (No integration is necessary.)

12.[R] How many steradians does one face of a cube subtend at

(a) One of the four vertices not on that face?

(b) The center of the cube? NOTE: No integration is necessary.

13.[M] In Example 2 the integral $\oint_C \widehat{\mathbf{r}} \cdot \mathbf{n}/r \, ds$ turned out to be 0. How would you explain this in terms of subtended angles?

14.[M] Let $\mathbf{F}$ and $\mathbf{G}$ be central vector fields in the plane with different centers.

(a) Show that the vector field $\mathbf{F}+\mathbf{G}$ is not a central field.

(b) Show that the divergence of $\mathbf{F}+\mathbf{G}$ is 0.

15.[M] In Example 6, we evaluated a surface integral by interpreting it in terms of the size of a subtended solid angle. Evaluate the integral directly, without that knowledge.

16.[M] Let $\mathcal{S}$ be the triangle whose vertices are $(1,0,0)$, $(0,1,0)$, $(0,0,1)$. Evaluate $\int_{\mathcal{S}} \frac{\widehat{\mathbf{r}}\cdot\mathbf{n}}{r^2}\, dS$ by using steradians.

17.[M] Evaluate the integral in Exercise 16 directly.

18.[M] Let $\mathbf{F}(x,y,z) = \frac{x\mathbf{i}+y\mathbf{j}+0\mathbf{k}}{x^2+y^2}$ be a vector field in space.

(a) What is the domain of $\mathbf{F}$?

(b) Sketch $\mathbf{F}(1,1,0)$ and $\mathbf{F}(1,1,2)$ with tails at the given points.

(c) Show $\mathbf{F}$ is not a central field.

(d) Show its divergence is 0.

Exercises 19 to 26 are related.

19.[M] Let $\mathbf{F}$ be a planar central field. Show that $\nabla \times \mathbf{F}$ is $\mathbf{0}$. HINT: $\mathbf{F}(x,y) = \frac{g(\sqrt{x^2+y^2}(x\mathbf{i}+y\mathbf{j}))}{\sqrt{x^2+y^2}}$ for some scalar function g.

20.[M] (This continues Exercise 19.) Show that $\mathbf{F}$ is a gradient field; to be specific, $\mathbf{F} = \nabla g(\sqrt{x^2+y^2})$.

21.[C] Carry out the computation to show that the *only* central fields in space that have zero divergence have the form $\mathbf{F}(\mathbf{r}) = c\widehat{\mathbf{r}}/r^2$, if the origin of the coordinates is at the center of the field.

22.[C] If we worked in four-dimensional space instead of the two-dimensional plane or three-dimensional space, which central fields do you think would have zero divergence? Carry out the calculation to confirm your conjecture.

23.[C] Let $\mathbf{F} = \widehat{\mathbf{r}}/r^2$ and S be the surface of the lopsided pyramid with square base, whose vertices are $(0,0,0)$, $(1,1,0)$, $(0,1,0)$, $(0,1,1)$, $(1,1,1)$.

(a) Sketch the pyramid.

(b) What is the integral of $\mathbf{F}\cdot\mathbf{n}$ over the square base?

(c) What is the integral of $\mathbf{F}\cdot\mathbf{n}$ over each of the remaining four faces?

(d) Evaluate $\oint_S \mathbf{F}\cdot\mathbf{n}\, dS$.

24.[C] Let C be the circle $x^2+y^2=4$ in the xy-plane. For each point Q in the disk bounded by C consider the central field with center Q, $\mathbf{F}(P) = \overrightarrow{PQ}/\|PQ\|^2$. Its magnitude is inversely proportional to the first power of the distance P is from Q. For each point Q consider the flux of $\mathbf{F}$ across C.

(a) Evaluate directly the flux when Q is the origin $(0,0)$.

(b) If Q is not the origin, evaluate the flux of $\mathbf{F}$.

(c) Evaluate the flux when Q lies on C.

25.[C] Let $\mathbf{F}$ be the central field in the plane, with center at $(1,0)$ and with magnitude inversely proportional to the first power of the distance to $(1,0)$: $\mathbf{F}(x,y) = \frac{(x-1)\mathbf{i}+y\mathbf{j}}{\|(x-1)\mathbf{i}+y\mathbf{j}\|^2}$. Let C be the circle of radius 2 and center at $(0,0)$.

(a) By thinking in terms of subtended angle, evaluate the flux $\oint_C \mathbf{F}\cdot\mathbf{n}\,ds$.

(b) Evaluate the flux by carrying out the integration.

26.[C] This exercise gives a geometric way to see why a central force is conservative. Let $\mathbf{F}(x,y) = f(r)\widehat{\mathbf{r}}$. Figure 18.4.12 shows $\mathbf{F}(x,y)$ and a short vector $\overrightarrow{d\mathbf{r}}$ and two circles.

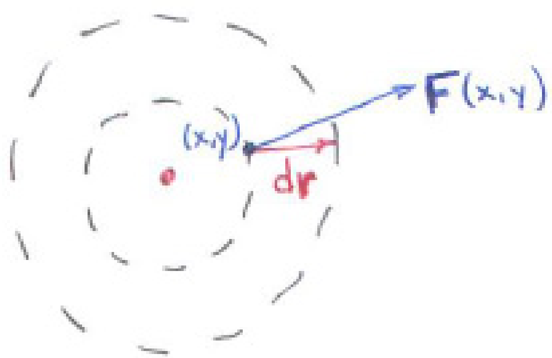


Figure 18.4.12:

(a) Why is $\mathbf{F}(x,y)\cdot\, d\mathbf{r}$ approximately $f(r)\,dr$, where dr is the difference in the radii of the two circles?

(b) Let C be a curve from A to B, where $A=(a,\alpha)$ and $B=(b,\beta)$ in polar coordinates. Why is $\int_C \mathbf{F}\cdot\ d\mathbf{r} = \int_a^b f(r)\,dr$?

(c) Why is $\mathbf{F}$ conservative?

Skill Drill

27.[R] Show that the derivative of $\frac{1}{3}\tan^3(x) - \tan(x) +$ x is $\tan^4(x)$.

28.[R] Use integration by parts to show that

$$\int \tan^n(x)\,dx = \frac{\tan^{n-1}(x)}{n-1} - \int \tan^{n-2}(x)\,dx.$$

29.[R] Entry 16 in the Table of Antiderivatives in the front cover of this book is:

$$\int \frac{dx}{x(ax+b} = \frac{1}{b}\ln\left|\frac{x}{ax+b}\right|.$$

(a) Use a partial fraction expansion to evaluate the antiderivative.

(b) Use differentiation to check that this formula is correct.

30.[R] Repeat Exercise 29 for entry 17 in the Table of Antiderivatives:

$$\int \frac{dx}{x(ax+b)} = \frac{1}{b}\ln\left|\frac{x}{ax+b}\right|.$$

31.[R] Show that $x\arccos(x) - \sqrt{1-x^2}$ is an integral of $\arccos(x)$.

32.[R] Find $\int \arctan(x)$.

33.[R]

(a) Find $\int xe^{ax}\,dx$.

(b) Use integration by parts to show that

$$\int x^m e^{ax}\,dx = \frac{x^m e^{ax}}{a} - \frac{m}{a}\int x^{m-1}e^{ax}\,dx.$$

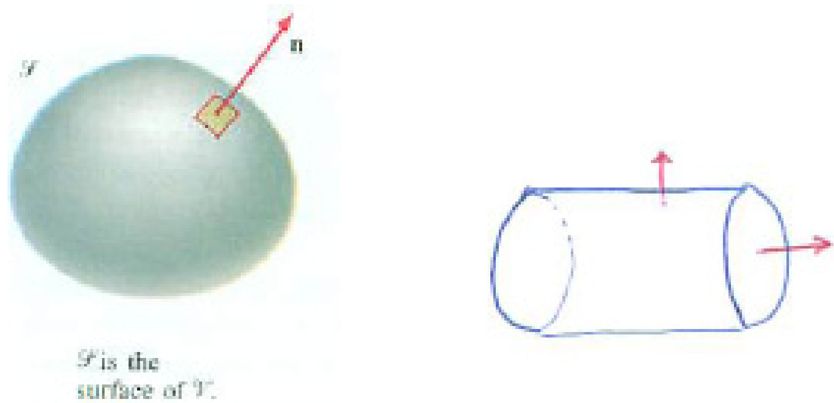
(c) Verify the equation in (b) by differentiating the right hand side.

18.5 The Divergence Theorem in Space (Gauss' Theorem)

In Sections 18.2 and 18.3 we developed Green's theorem and applied it in two forms for a vector field $\mathbf{F}$ in the plane. One form concerned the line integral of the tangential component of $\mathbf{F}$, $\oint_C \mathbf{F}\cdot\mathbf{T}\,ds$, also written as $\oint_C \mathbf{F}\cdot d\mathbf{r}$. The other concerned the integral of the normal component of $\mathbf{F}$, $\oint_C \mathbf{F}\cdot\mathbf{n}\,ds$. In this section we develop the **Divergence Theorem**, an extension of the second form from the plane to space. The extension of the first form to space is the subject of Section 18.6. In Section 18.7 the Divergence Theorem will be applied to electro-magnetism.

The Divergence (or Gauss's) Theorem

Consider a region $\mathcal{R}$ in space bounded by a surface $\mathcal{S}$. For instance, $\mathcal{R}$ may be a ball and $\mathcal{S}$ its surface. This is a case encountered in the elementary theory of electro-magnetism. In another case, $\mathcal{R}$ is a right circular cylinder and $\mathcal{S}$ is its surface, which consists of two disks and its curved side. See Figure 18.5.1(a). Both figures show typical unit exterior normals, perpendicular to the surface.



(a) (b)

Figure 18.5.1:

The Divergence Theorem relates an integral over the surface to an integral over the region it bounds.

Theorem (Divergence Theorem —One-Surface Case.). *Let $\mathcal{V}$ be the region in space bounded by the surface $\mathcal{S}$. Let $\mathbf{n}$ denote the exterior unit normal of $\mathcal{V}$*

along the boundary $\mathcal{S}$. Then

$$\int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{n}\; ds = \int_{\mathcal{V}} \nabla\cdot\mathbf{F}\; dV$$

for any vector field $\mathbf{F}$ *defined on* $\mathcal{V}$*.*

State the Theorem aloud.

In words: "The integral of the normal component of $\mathbf{F}$ over a surface equals the integral of the divergence of $\mathbf{F}$ over the region the surface bounds."

The integral $\int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{n}\; dS$ is called the **flux** of the field $\mathbf{F}$ across the surface $\mathcal{S}$.

If $\mathbf{F} = P\mathbf{i}+Q\mathbf{j}+R\mathbf{k}$ and $\cos(\alpha)$, $\cos(\beta)$, and $\cos(\gamma)$ are the direction cosines of the exterior normal, then the Divergence Theorem reads

$$\int_{\mathcal{S}} (P\mathbf{i}+Q\mathbf{j}+R\mathbf{k})\cdot(\cos(\alpha)\mathbf{i}+\cos(\beta)\mathbf{j}+\cos(\gamma)\mathbf{k})\; dS = \int_{\mathcal{V}} \left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)\; dV.$$

Direction cosines are defined in Section 14.4.

Evaluating the dot product puts the Divergence Theorem in the form

$$\int_{\mathcal{S}} (P\cos(\alpha)+Q\cos(\beta)+R\cos(\gamma))\; dS = \int_{\mathcal{V}} \left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)\; dV.$$

When the Divergence Theorem is expressed in this form, we see that it amounts to three scalar theorems:

$$\int_{\mathcal{S}} P\cos(\alpha)\; dS = \int_{\mathcal{V}} \frac{\partial P}{\partial x}\; dV, \quad \int_{\mathcal{S}} Q\cos(\beta)\; dS = \int_{\mathcal{V}} \frac{\partial Q}{\partial y}\; dV, \quad \text{and} \quad \int_{\mathcal{S}} R\cos(\gamma)\; dS = \tag{18.5.1}$$

As is to be expected, establishing these three equations proves the Divergence Theorem. We delay the proof to the end of this section, after we have shown how the Divergence Theorem is applied.

You could have guessed the result in this Example by thinking in terms of the solid angle and steradians. Why?

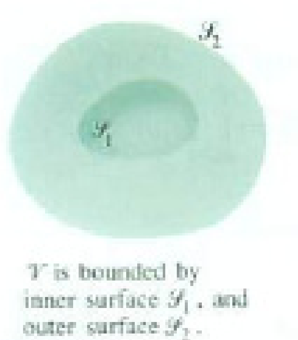


Figure 18.5.2:

Two-Surface Version of the Divergence Theorem

The Divergence Theorem also holds if the solid region has several holes like a piece of Swiss cheese. In this case, the boundary consists of several separate closed surfaces. The most important case is when there is just one hole and hence an inner surface $\mathcal{S}_1$ and an outer surface $\mathcal{S}_2$ as shown in Figure 18.5.2.

Theorem (Divergence Theorem — Two-Surface Case.). *Let $\mathcal{V}$ be a region in space bounded by the surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$. Let $\mathbf{n}^*$ denote the exterior normal along the boundary. Then*

$$\int\limits_{\mathcal{S}_1} \mathbf{F}\cdot\mathbf{n}^*\ dS + \int\limits_{\mathcal{S}_2} \mathbf{F}\cdot\mathbf{n}^*\ dS = \int\limits_{\mathcal{V}} \operatorname{div}\mathbf{F}\ dV$$

for any vector field defined on $\mathcal{V}$.

Compare with (18.2.4) in Exercise 3 in Section 18.2.

The importance of this form of the Divergence Theorem is that it allows us to conclude that the flux across each of the surfaces are the same provided these surfaces form the boundary of a solid where $\operatorname{div}\mathbf{F} = 0$.

Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two closed surfaces that form the boundary of the region $\mathcal{V}$. Let $\mathbf{F}$ be a vector field defined on $\mathcal{V}$ such that the divergence of $\mathbf{F}$, $\nabla\cdot\mathbf{F}$, is 0 throughout $\mathcal{V}$. Then

$$\int\limits_{\mathcal{S}_1} \mathbf{F}\cdot\mathbf{n}\ dS = \int\limits_{\mathcal{S}_2} \mathbf{F}\cdot\mathbf{n}\ dS \tag{18.5.2}$$

The proof of this result closely parallels the derivation of (18.2.4) in Section 18.2.

The next example is a major application of (18.5.2), which enables us, if the divergence of $\mathbf{F}$ is 0, to replace the integral of $\mathbf{F}\cdot\mathbf{n}$ over a surface by an integral of $\mathbf{F}\cdot\mathbf{n}$ over a more convenient surface.

EXAMPLE 1 Let $\mathbf{F}(\mathbf{r}) = \widehat{\mathbf{r}}/r^2$, the inverse square vector field with center at the origin. Let $\mathcal{S}$ be a convex surface that encloses the origin. Find the flux of $\mathbf{F}$ over the surface, $\int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{n}\ dS$.

SOLUTION Select a sphere with center at the origin that does not intersect $\mathcal{S}$. This sphere should be very small in order to miss $\mathcal{S}$. Call this spherical surface $\mathcal{S}_1$ and its radius a. Then, by (18.5.2),

$$\int\limits_{\mathcal{S}} \mathbf{F}\cdot\mathbf{n}\ dS = \int\limits_{\mathcal{S}_1} \mathbf{F}\cdot\mathbf{n}\ dS$$

But $\int_{\mathcal{S}_1} \mathbf{F}\cdot\mathbf{n}\ dS$ is easy because the integrand $(\widehat{\mathbf{r}}/r^2)\cdot\mathbf{n}$ equals $\frac{\mathbf{r}\cdot\mathbf{n}}{r^2}$. Then, $\mathbf{r}\cdot\mathbf{n}$ is just 1. Thus:

$$\int\limits_{\mathcal{S}_1} \mathbf{F}\cdot\mathbf{n}\ dS = \int\limits_{\mathcal{S}_1} \frac{1}{a^2}\ dS = \frac{1}{a^2}\int\limits_{\mathcal{S}_1}\ dS = \frac{1}{a^2}4\pi a^2 = 4\pi.$$

◇

A **uniform** or **constant** vector field is a vector field where vectors at every point are all identical. Such fields are used in the next example.

EXAMPLE 2 Verify the Divergence Theorem for the constant field $\mathbf{F}(x, y, z) = 2\mathbf{i}+3\mathbf{j}+4\mathbf{k}$ and the surface $\mathcal{S}$ of a cube whose sides have length 5 and is situated as shown in Figure 18.5.3.

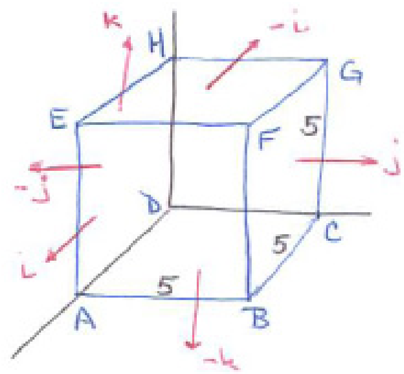


Figure 18.5.3:

SOLUTION To find $\int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{n}\, dS$ we consider the integral of $\mathbf{F}\cdot\mathbf{n}$ over each of the six faces.

On the bottom face, $ABCD$ the unit exterior normal is $-\mathbf{k}$. Thus

$$\mathbf{F}\cdot\mathbf{n} = (2\mathbf{i}+3\mathbf{j}+4\mathbf{k})\cdot(-\mathbf{k}) = -4.$$

So

$$\int\limits_{ABCD} \mathbf{F}\cdot\mathbf{n}\, dS = \int\limits_{ABCD} (-4)\; dS = -4\int\limits_{ABCD} dS = (-4)(25) = -100.$$

The integral over the top face involves the exterior unit normal $\mathbf{k}$ instead of $-\mathbf{k}$. Then $\int_{EFGH} \mathbf{F}\cdot\mathbf{n}\, dS = 100$. The sum of these two integrals is 0. Similar computations show that the flux of $\mathbf{F}$ over the entire surface is 0.

The Divergence Theorem says that this flux equals $\int_{\mathcal{R}} \operatorname{div}\mathbf{F}\, dV$, where $\mathcal{R}$ is the solid cube. Now, $\operatorname{div}\mathbf{F} = \partial(2)/\partial x + \partial(3)/\partial y + \partial(4)/\partial z = 0+0+0 = 0$. So the integral of $\operatorname{div}\mathbf{F}$ over $\mathcal{R}$ is 0, verifying the divergence theorem. ◇

Why div $\mathbf{F}$ is Called the Divergence

Let $\mathbf{F}(x, y, z)$ be the vector field describing the **flow** for a gas. That is, $\mathbf{F}(x, y, z)$ is the product of the density of the gas at (x, y, z) and the velocity vector of the gas there.

The integral $\int_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n}\, dS$ over a closed surface $\mathcal{S}$ represents the tendency of the gas to leave the region $\mathcal{R}$ that $\mathcal{S}$ bounds. If that integral is positive the gas is tending to escape or "diverge". If negative, the net effect is for the amount of gas in $\mathcal{R}$ to increase and become denser.

Let $\rho(x, y, z, t)$ be the density of the gas at time t at the point P, with units mass per unit volume. Then $\int_{\mathcal{R}} \rho\, dV$ is the total mass of gas in $\mathcal{R}$ at a given time. So the rate at which the mass in $\mathcal{R}$ changes is given by the derivative

$$\frac{d}{dt} \int\limits_{\mathcal{R}} \rho\, dV.$$

If ρ is sufficiently well-behaved, mathematicians assure us that we may "differentiate past the integral sign." Then

$$\frac{d}{dt} \int\limits_{\mathcal{R}} \rho\, dV = \int\limits_{\mathcal{R}} \frac{\partial p}{\partial t}\, dV.$$

Therefore

$$\int\limits_{\mathcal{R}} \frac{\partial p}{\partial t}\, dV = \int\limits_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n}\, dS$$

since both represent the rate at which gas accumulates in or escapes from $\mathcal{R}$. But, by the Divergence Theorem, $\int_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n}\, dS = \int_{\mathcal{R}} \nabla \cdot \mathbf{F}\, dV$, and so

$$\int\limits_{\mathcal{R}} \nabla \cdot \mathbf{F}\, dV = \int\limits_{\mathcal{R}} \frac{\partial p}{\partial t}\, dV$$

or,

$$\int\limits_{\mathcal{R}} (\nabla \cdot \mathbf{F} - \frac{\partial p}{\partial t})\, dV = 0. \qquad (18.5.3)$$

From this is it possible to conclude that $\nabla \cdot \mathbf{F} - \frac{\partial p}{\partial t} = 0$?

Recall that the Zero-Integral Principle (see Section 6.3) says: If a continuous function f on an interval $[a, b]$ has the property that $\int_c^d f(x)\, dx = 0$ for every subinterval $[c, d]$ then $f(x) = 0$ on $[a, b]$. A natural extention of the Zero-Integral Principle (see Exercise 27) is:

Zero-Integral Principle in Space

Let $\mathcal{R}$ be a region in space, that is, a set of points in space that is bounded by a surface, and let f be a continuous function on $\mathcal{R}$. Assume that for every region $\mathcal{S}$ in $\mathcal{R}$, $\int_{\mathcal{S}} f(P)\, dS = 0$. Then $f(P) = 0$ for all P in $\mathcal{R}$.

Equation 18.5.3 holds not just for the solid $\mathcal{R}$ but for any solid region within $\mathcal{R}$. By the Zero-Integral Principle in Space, the integrand must be zero thoughout $\mathcal{R}$, and we conclude that

$$\nabla \cdot \mathbf{F} = \frac{\partial p}{\partial t}.$$

This equation tells us that $\operatorname{div} \mathbf{F}$ at a point P represents the rate gas is getting denser or lighter near P. That is why $\operatorname{div} \mathbf{F}$ is called the "divergence of $\mathbf{F}$". Where $\operatorname{div} \mathbf{F}$ is positive, the gas is dissipating. Where $\operatorname{div} \mathbf{F}$ is negative, the gas is collecting.

See Exercise 20 in Section 18.3.

For this reason a vector field for which the divergence is 0 is called **incompressible**. An incompressible is also called "divergence free".

We conclude this section with a proof of the Divergence Theorem.

Proof of the Divergence Theorem

We prove the theorem only for the special case that each line parallel to an axis meets the surface $\mathcal{S}$ in at most two points and $\mathcal{V}$ is convex. We prove the third equation in (18.5.1). The other two are established in the same way.

We wish to show that

$$\int_{\mathcal{V}} R\cos(\gamma)\ dS = \int_{\mathcal{V}} \frac{\partial R}{\partial z}\ dV. \tag{18.5.4}$$

Let $\mathcal{A}$ be the projection of $\mathcal{S}$ on the xy plane. Its description is

$$a \leq x \leq b, \qquad y_1(x) \leq y \leq y_2(x).$$

The description of $\mathcal{V}$ is then

$$a \leq x \leq b, \qquad y_1(x) \leq y \leq y_2(x), \qquad z_1(x,y) \leq z \leq z_2(x,y).$$

Then (see Figure 18.5.4)

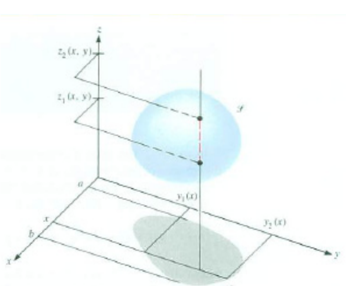


Figure 18.5.4:

$$\int_{\mathcal{V}} \frac{\partial R}{\partial z}\ dV = \int_a^b \int_{y_1(x)}^{y_2(x)} \int_{z_1(x,y)}^{z_2(x,y)} \frac{\partial R}{\partial z}\ dz\ dy\ dx. \tag{18.5.5}$$

The first integration gives

$$\int_{z_1(x,y)}^{z_2(x,y)} \frac{\partial R}{\partial z}\ dz = R(x,y,z_2) - R(x,y,z_1),$$

by the Fundamental Theorem of Calculus. We have, therefore,

$$\int\limits_{\mathcal{V}} \frac{\partial R}{\partial z}\ dV = \int\limits_{a}^{b} \int\limits_{y_1(x)}^{y_2(x)} \left(R(x,y,z_2) - R(x,y,z_1)\right)\ dy\ dx,$$

hence

$$\int\limits_{\mathcal{V}} \frac{\partial R}{\partial z}\ dV = \int\limits_{\mathcal{A}} \left(R(x,y,z_2) - R(x,y,z_1)\right)\ dA.$$

This says that, essentially, on the "top half" of $\mathcal{V}$, where $0 < \gamma < \pi/2$, $dA = \cos(\gamma)\ dS$ is positive. And, on the bottom half of $\mathcal{V}$, where $\pi/2 < \gamma < \pi$, $dA = -\cos(\gamma)\ dS$. According to (17.7.1) in Section 17.7, the last integral equals

$$\int\limits_{\mathcal{S}} R(x,y,z)\cos(\gamma)\ dS.$$

Thus

$$\int\limits_{\mathcal{V}} \frac{\partial R}{\partial z}\ dV = \int\limits_{\mathcal{S}} R\cos\gamma\ dS,$$

and (18.5.4) is established.

Similar arguments establish the other two equations in (18.5.1).

Summary

We stated the Divergence Theorem for a single surface and for two surfaces. They enable one to calculate the flux of a vector field $\mathbf{F}$ in terms of an integral of its divergence $\nabla \cdot \mathbf{F}$ over a region. This is especially useful for fields that are incompressible (divergence free). The most famous such field in space is the inverse-square vector field: $\widehat{\mathbf{r}}/r^2$. The flux across a surface of such a field depends on whether its center is inside or outside the surface. Specifically, if the center is at Q and the field is of the form $c\frac{\overrightarrow{QP}}{\|\overrightarrow{QP}\|^3}$, its flux across a surface not enclosing Q is 0. If it encloses Q, its flux is 4π. This is a consequence of the divergence theorem. It also can be explained geometrically, in terms of solid angles.

EXERCISES for Section 18.5 *Key:* R–routine, M–moderate, C–challenging

1.[R] State the Divergence Theorem in symbols.

2.[R] State the Divergence Theorem using only words, not using symbols, such as $\mathbf{F}$, $\nabla\cdot\mathbf{F}$, $\mathbf{n}$, $\mathcal{S}$, or $\mathcal{V}$.

3.[R] Explain why $\nabla\cdot\mathbf{F}$ at a point P can be expressed as a coordinate-free limit.

4.[R] What is the two-surface version of Gauss's theorem?

5.[R] Verify the divergence theorem for $\mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + 0\mathbf{k}$ and the surface $x^2+y^2+z^2 = 9$.

6.[R] Verify the divergence theorem for the field $\mathbf{F}(x,y,z) = x\mathbf{i}$ and the cube whose vertices are $(0,0,0)$, $(2,0,0)$, $(2,2,0)$, $(0,2,0)$, $(0,0,2)$, $(2,0,2)$, $(2,2,2)$, $(0,2,2)$.

7.[R] Verify the divergence theorem for $\mathbf{F} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ and the tetrahedron whose four vertices are $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.

8.[R] Verify the two-surface version of Gauss's theorem for $\mathbf{F}(x,y,z) = (x^2+y^2+z^2)(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})$ and the surfaces are the spheres of radii 2 and 3 centered at the origin.

9.[R] Let $\mathbf{F} = 2x\mathbf{i} + 3y\mathbf{j} + (5z+6x)\mathbf{k}$, and let $\mathbf{G} = (2x+4z^2)\mathbf{i} + (3y+5x)\mathbf{j} + 5z\mathbf{k}$. Show that

$$\int\limits_{\mathcal{S}} \mathbf{F}\cdot\mathbf{n}\,dS = \int\limits_{\mathcal{S}} \mathbf{G}\cdot\mathbf{n}\,dS,$$

where $\mathcal{S}$ is any surface bounding a region in space.

10.[R] Show that the divergence of $\widehat{\mathbf{r}}/r^2$ is 0. HINT: $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

In Exercises 11 to 18 use the Divergence Theorem.

11.[R] Let $\mathcal{V}$ be the solid region bounded by the xy plane and the paraboloid $z = 9 - x^2 - y^2$. Evaluate $\int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{n}\,dS$, where $\mathbf{F} = y^3\mathbf{i} + z^3\mathbf{j} + x^3\mathbf{k}$ and $\mathcal{S}$ is the boundary of $\mathcal{V}$.

12.[R] Evaluate $\int_{\mathcal{V}} \nabla\cdot\mathbf{F}\,dV$ for $\mathbf{F} = \sqrt{x^2+y^2+z^2}(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})$ and $\mathcal{V}$ the ball of radius 2 and center at $(0,0,0)$.

In Exercises 13 and 14 find $\int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{n}\,dS$ for the given $\mathbf{F}$ and $\mathcal{S}$.

13.[R] $\mathbf{F} = z\sqrt{x^2+z^2}\mathbf{i} + (y+3)\mathbf{j} - x\sqrt{x^2+z^2}\mathbf{k}$ and $\mathcal{S}$ is the boundary of the solid region between $z = x^2+y^2$ and the plane $z = 4x$.

14.[R] $\mathbf{F} = x\mathbf{i} + (3y + z)\mathbf{j} + (4x+2z)\mathbf{k}$ and $\mathcal{S}$ is the surface of the cube bounded by the planes $x=1$, $x=3$, $y=2$, $y=4$, $z=3$ and $z=5$.

15.[R] Evaluate $\int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{n}\,dS$, where $\mathbf{F} = 4xz\mathbf{i} - y^2\mathbf{j} + yz\mathbf{k}$ and $\mathcal{S}$ is the surface of the cube bounded by the planes $x=0$, $x=1$, $y=0$, $z=0$, and $z=1$, with the face corresponding to $x=1$ removed.

16.[R] Evaluate $\int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{n}\,dS$, where $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + 2x\mathbf{k}$ and $\mathcal{S}$ is the boundary of the tetrahedron with vertices $(1,2,3)$, $(1,0,1)$ $(2,1,4)$, and $(1,3,5)$.

17.[R] Let $\mathcal{S}$ be a surface of area S that bounds a region $\mathcal{V}$ of volume V. Assume that $\|\mathbf{F}(P)\| \le 5$ for all points P on the surface $\mathcal{S}$. What can be said about $\int_{\mathcal{V}} \nabla\cdot\mathbf{F}\,dV$?

18.[R] Evaluate $\int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{n}\,dS$, where $\mathbf{F} = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$ and $\mathcal{S}$ is the sphere of radius a and center $(0,0,0)$.

In Exercises 19 to 22 evaluate $\int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{n}\,dS$ for $\mathbf{F} = \widehat{\mathbf{r}}/r^2$ and the given surfaces, doing as little calculation as possible.

19.[R] $\mathcal{S}$ is the sphere of radius 2 and center $(5,3,1)$.

20.[R] $\mathcal{S}$ is the sphere of radius 3 and center $(1,0,1)$.

21.[R] $\mathcal{S}$ is the surface of the box bounded by the planes $x=-1$, $x=2$, $y=2$, $y=3$, $z=-1$, and $z=6$.

22.[R] $\mathcal{S}$ is the surface of the box bounded by the planes $x = -1$, $x = 2$, $y = -1$, $y = 3$, $z = -1$, and $z = 4$.

23.[M] Assume that the flux of $\mathbf{F}$ across every sphere is 0. Must the flux of $\mathbf{F}$ across the surface of every cube be 0 also?

24.[R] If $\mathbf{F}$ is always tangent to a given surface $\mathcal{S}$ what can be said about the integral of $\nabla \cdot \mathbf{F}$ over the region that $\mathcal{S}$ bounds?

25.[M] Let $\mathbf{F}(\mathbf{r}) = f(r)\widehat{\mathbf{r}}$ be a central vector field in space that has zero divergence. Show that $f(r)$ must have the form $f(r) = a/r^2$ for some constant a. HINT: Consider the flux of $\mathbf{F}$ across the closed surface in Figure 18.5.5.

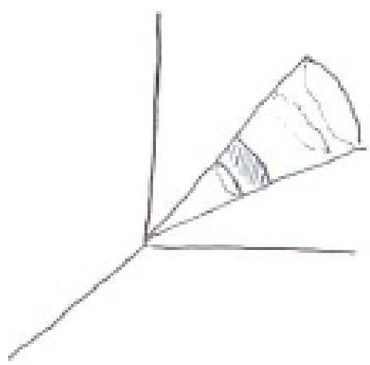


Figure 18.5.5:

26.[M] Let $\mathbf{F}$ be defined everywhere except at the origin and be divergence-free. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two closed surfaces that enclose the origin. Explain why $\int_{\mathcal{S}_1} \mathbf{F} \cdot \mathbf{n}\, dS = \int_{\mathcal{S}_2} \mathbf{F} \cdot \mathbf{n}\, dS$. (The two surfaces may intersect.)

27.[M] Provide the details for the proof of the Zero-Integral Principle in Space. HINT: You need to consider the two cases when $f > 0$ and $f < 0$.

28.[M] Show that the flux of an inverse-square central field $c\widehat{\mathbf{r}}/r^2$ across any closed surface that bounds a region that does not contain the origin is zero.

29.[C]

(a) Show that the proof in the text of the Divergence Theorem applies to a tetrahedron. HINT: Choose your coordinate system carefully.

(b) Deduce that if the Divergence Theorem holds for a tetrahedron then it holds for any polyhedron. HINT: Each polyhedron can be cut into tetrahedra.

30.[C] In Exercise 25 you were asked to show generally that the only central fields with zero divergence are the inverse square fields. Show this, instead, by computing the divergence of $\mathbf{F}(x, y, z) = f(r)\hat{r}$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

31.[C] Let $\mathbf{F}$ be defined everywhere in space except at the origin. Assume that

$$\lim_{\|\mathbf{r}\| \to \infty} \frac{\mathbf{F}(\mathbf{r})}{\|\mathbf{r}\|^2} = \mathbf{0}$$

and that $\mathbf{F}$ is defined everywhere except at the origin, and is divergence free. What can be said about $\int_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n}\, dS$, where $\mathcal{S}$ is the sphere of radius 2 centered at the origin?

We proved one-third of the Divergence Theorem. Exercises 32 and 33 concern the other two-thirds.

32.[C] Prove that

$$\int_{\mathcal{S}} Q \cos(\beta)\, dS = \int_{\mathcal{V}} \frac{\partial Q}{\partial y}\, dV.$$

33.[C] Prove that

$$\int_{\mathcal{S}} P \cos(\alpha)\, dS = \int_{\mathcal{V}} \frac{\partial P}{\partial x}\, dV.$$

34.[C] Let f be a scalar function $\mathbf{F}(x, y, z) = f(r)\widehat{\mathbf{r}}$, where $r = \|\mathbf{r}\}$ and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Show that if $\nabla \cdot \mathbf{F} = 0$, then $f(r) = c/r^2$ for some constant c.

18.6 Stokes' Theorem

In Section 18.1 we learned that Green's theorem in the xy-plane can be written as

$$\oint_C \mathbf{F} \cdot \, d\mathbf{r} = \int_{\mathcal{R}} (\mathbf{curl}\,\mathbf{F}) \cdot \mathbf{k}\, dA,$$

where C is counterclockwise and C bounds the region $\mathcal{R}$. The general Stokes' Theorem introduced in this section extends this result to closed curves in space. It asserts that if the closed curve C bounds a surface $\mathcal{S}$, then

$$\oint_C \mathbf{F} \cdot \, d\mathbf{r} = \int_{\mathcal{S}} (\mathbf{curl}\,\mathbf{F}) \cdot \mathbf{n}\, dS.$$

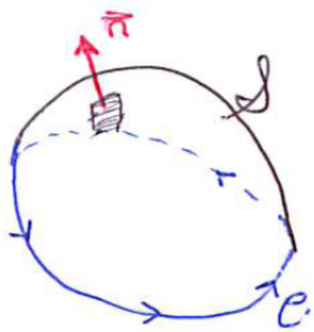


Figure 18.6.1:

As usual, the vector $\mathbf{n}$ is a unit normal to the surface. There are two such normals at each point on the surface. In a moment we describe how to decide which unit normal vector to use. The choice depends on the orientation of C.

In words, Stokes' theorem reads, "The circulation of a vector field around a closed curve is equal to the integral of the normal component of the curl of the field over any surface that the curve bounds."

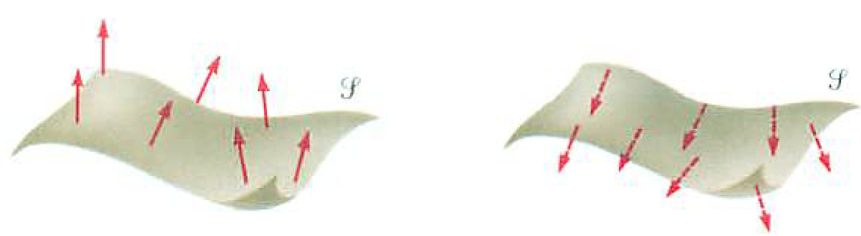
Stokes' published his theorem in 1854 (without proof, for it appeared as a question on a Cambridge University examination). By 1870 it was in common use. It is the most recent of the three major theorems discussed in this chapter, for Green published his theorem in 1828 and Gauss published the divergence theorem in 1839.

Figure 18.6.2:

Choosing the Normal n

In order to state Stokes' theorem precisely, we must describe what kind of surface $\mathcal{S}$ is permitted and which of the two possible normals $\mathbf{n}$ to choose.

The typical surfaces $\mathcal{S}$ that we consider have the property that it is possible to assign, at each point on $\mathcal{S}$, a unit normal $\mathbf{n}$ in a continuous manner. On the surface shown in Figure 18.6.2, there are two ways to do this. They are shown in Figure 18.6.3.



(a) (b)

Figure 18.6.3:

But, for the surface shown in Figure 18.6.4 (a Möbius band), it is impossible to make such a choice. If you start with choice (1) and move the normal continuously along the surface, by the time you return to the initial point on the surface at stage (9), you have the opposite normal. A surface for which a continuous choice *can* be made is called **orientable** or **two-sided**. Stokes' theorem holds for orientable surfaces, which include, for instance, any part of the surface of a convex body, such as a ball, cube or cylinder.

Right-hand rule for choosing $\mathbf{n}$.

Consider an orientable surface $\mathcal{S}$, bounded by a parameterized curve C so that the curve is swept out in a definite direction. If the surface is part of a plane, we can simply use the right-hand rule to choose $\mathbf{n}$: The direction of $\mathbf{n}$ should match the thumb of the right hand if the fingers curl in the direction of C and the thumb and palm are perpendicular to the plane. If the surface is not flat, we still use the right-hand rule to choose a normal at points near C. The choice of one normal determines normals throughout the surface. Figure 18.6.5 illustrates the choice of $\mathbf{n}$. For instance, if C is counterclockwise in the xy-plane, this definition picks out the normal $\mathbf{k}$, not $-\mathbf{k}$.

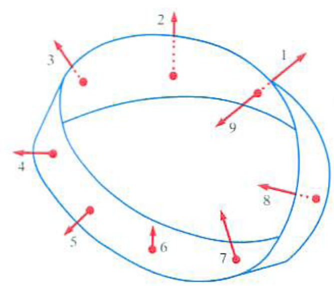
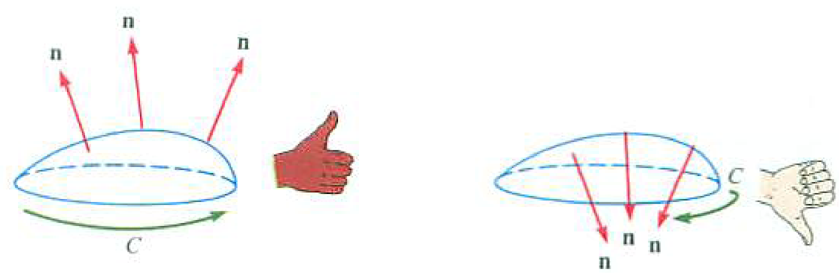


Figure 18.6.4: Follow the choices through all nine stages — there's trouble.



(a) (b)

Figure 18.6.5:

Theorem 18.6.1 (Stokes' theorem)**.** *Let $\mathcal{S}$ be an orientable surface bounded by the parameterized curve C. At each point of $\mathcal{S}$ let $\mathbf{n}$ be the unit normal chosen by the right-hand rule. Let $\mathbf{F}$ be a vector field defined on some region in space including $\mathcal{S}$. Then*

$$\oint_C \mathbf{F}\cdot\, d\mathbf{r} = \int_{\mathcal{S}} (\nabla\times\mathbf{F})\cdot\mathbf{n}\; dS. \tag{18.6.1}$$

Some Applications of Stokes' Theorem

Stokes' theorem enables us to replace $\int_{\mathcal{S}}(\mathbf{curl}\,\mathbf{F})\cdot\mathbf{n}\; dS$ by a similar integral over a surface that might be simpler than $\mathcal{S}$. That is the substance of the following special case of Stokes' theorem.

One way to evaluate some surface integrals is to choose a simpler surface.

Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two surfaces bounded by the same curve C and oriented so that they yield the same orientation on C. Let $\mathbf{F}$ be a vector field defined on both $\mathcal{S}_1$ and $\mathcal{S}_2$. Then

$$\int_{\mathcal{S}_1} (\mathbf{curl}\,\mathbf{F}) \cdot \mathbf{n}\; dS = \int_{\mathcal{S}_2} (\mathbf{curl}\,\mathbf{F}) \cdot \mathbf{n}\; dS \tag{18.6.2}$$

The two integrals in (18.6.2) are equal since both equal $\oint_C \mathbf{F} \cdot\, d\mathbf{r}$.

EXAMPLE 1 Let $\mathbf{F} = xe^z\mathbf{i} + (x + xz)\mathbf{j} + 3e^z\mathbf{k}$ and let $\mathcal{S}$ be the top half of the sphere $x^2 + y^2 + z^2 = 1$. Find $\int_{\mathcal{S}} (\mathbf{curl}\,\mathbf{F}) \cdot \mathbf{n}\; dS$, where $\mathbf{n}$ is the outward normal. (See Figure 18.6.6.)

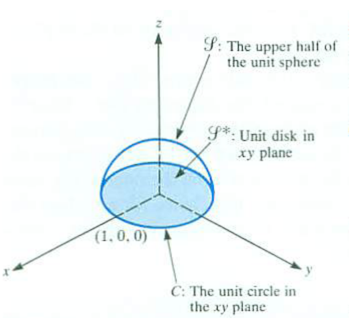


Figure 18.6.6: ARTIST: Add an arrow to indicate the unit circle in the plane is to be oriented counterclockwise. Also add "counterclockwise" to the text label for C.

SOLUTION Let $\mathcal{S}^*$ be the flat base of the hemisphere. By (18.6.2),

$$\int_{\mathcal{S}} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\; dS = \int_{\mathcal{S}^*} (\nabla \times \mathbf{F}) \cdot \mathbf{k}\; dS.$$

(On $\mathcal{S}^*$ note that $\mathbf{k}$, not $-\mathbf{k}$, is the correct normal to use.)

A straightforward calculation shows that

$$\nabla \times \mathbf{F} = -x\mathbf{i} + xe^z\mathbf{j} + (z+1)\mathbf{k},$$

hence $(\nabla \times \mathbf{F}) \cdot \mathbf{k} = z + 1$. On $\mathcal{S}^*$, $z = 0$, so

$$\int_{\mathcal{S}^*} (\nabla \times \mathbf{F}) \cdot \mathbf{k}\; dS = \int_{\mathcal{S}^*} dS = \pi.$$

thus the original integral over $\mathcal{S}$ is also π. ⋄

Just as there are two-curve versions of Green's Theorem and of the Divergence Theorem, there is a two-curve version of Stokes' Theorem.

Two-curve version of Stokes's Theorem

Stokes' Theorem for a Surface Bounded by Two Closed Curves

Let $\mathcal{S}$ be an orientable surface whose boundary consists of the two closed curves C_1 and C_2. Give C_1 an orientation. Orient $\mathcal{S}$ consistent with the the right-hand rule, as applied to C_1. Give C_2 the same orientation as C_1. (If C_2 is moved on $\mathcal{S}$ to C_1, the orientations will agree.) Then

$$\oint_{C_1} \mathbf{F} \cdot\, d\mathbf{r} - \oint_{C_2} \mathbf{F} \cdot\, d\mathbf{r} = \int_{\mathcal{S}} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\; dS. \tag{18.6.3}$$

Proof

Figure 18.6.7(a) shows the typical situation.

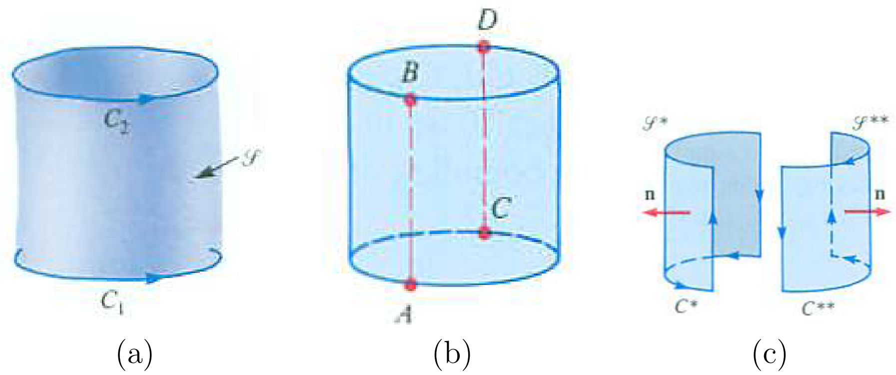


Figure 18.6.7:

The cancellation principle was introduced in Section 18.2.

We will obtain (18.6.3) from Stokes's theorem with the aid of the cancellation principle. Introduce lines AB and CD on $\mathcal{S}$, cutting $\mathcal{S}$ into two surfaces, $\mathcal{S}^*$ and $\mathcal{S}^{**}$. (See Figure 18.6.7(c).) Now apply Stokes's theorem to $\mathcal{S}^*$ and $\mathcal{S}^{**}$. (See Figure 18.6.7(c).)

Let C^* be the curve that bounds $\mathcal{S}^*$, oriented so that where it overlaps C_1 it has the same orientation as C_1. Let C^{**} be the curve that bounds $\mathcal{S}^{**}$, again oriented to match C_1. (See Figure 18.6.7(c).)

By Stokes' theorem,

$$\oint_{C^*} \mathbf{F} \cdot \, d\mathbf{r} = \int_{\mathcal{S}^*} (\mathbf{curl}\,\mathbf{F}) \cdot \mathbf{n}\; dS \tag{18.6.4}$$

and

$$\oint_{C^{**}} \mathbf{F} \cdot \, d\mathbf{r} = \int_{\mathcal{S}^{**}} (\mathbf{curl}\,\mathbf{F}) \cdot \mathbf{n}\; dS. \tag{18.6.5}$$

Adding (18.6.4) and (18.6.5) and using the cancellation principle gives

$$\oint_{C_1} \mathbf{F} \cdot \, d\mathbf{r} - \oint_{C_2} \mathbf{F} \cdot \, d\mathbf{r} = \int_{\mathcal{S}} (\mathbf{curl}\,\mathbf{F}) \cdot \mathbf{n}\; dS.$$

•

Recall, from Section 18.2, that $\mathbf{F}$ is irrotational when $\mathbf{curl}\,\mathbf{F} = \mathbf{0}$.

In practice, it is most common to apply (18.6.3) when $\mathbf{curl}\,\mathbf{F} = \mathbf{0}$. This is so important we state it explicitly:

Let $\mathbf{F}$ be a field such that $\mathbf{curl}\,\mathbf{F} = \mathbf{0}$. Let C_1 and C_2 be two closed curves that together bound an orientable surface $\mathcal{S}$ on which $\mathbf{F}$ is defined. If C_1 and C_2 are similarly oriented, then

$$\oint\limits_{C_1} \mathbf{F}\cdot\, d\mathbf{r} = \oint\limits_{C_2} \mathbf{F}\cdot\, d\mathbf{r}. \tag{18.6.6}$$

Equation (18.6.6) follows directly from (18.6.3) since $\int_{\mathcal{S}} (\mathbf{curl}\,\mathbf{F})\cdot\mathbf{n}\,dS = 0$.

EXAMPLE 2 Assume that $\mathbf{F}$ is irrotational and defined everywhere except on the z-axis. Given that $\oint_{C_1} \mathbf{F}\cdot\, d\mathbf{r} = 3$, find (a) $\oint_{C_2} \mathbf{F}\cdot\, d\mathbf{r}$ and (b) $\oint_{C_3} \mathbf{F}\cdot\, d\mathbf{r}$. (See Figure 18.6.8.)

SOLUTION (a) By (18.6.6), $\oint_{C_2} \mathbf{F}\cdot\, d\mathbf{r} = \oint_{C_1} \mathbf{F}\cdot\, d\mathbf{r} = 3$. (b) By Stokes' theorem, (18.6.1), $\oint_{C_3} \mathbf{F}\cdot\, d\mathbf{r} = 0$. ◇

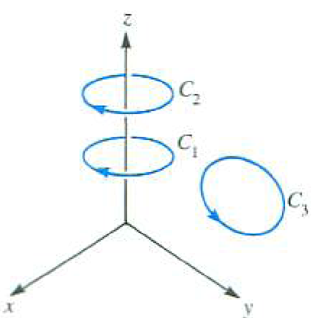


Figure 18.6.8:

Curl and Conservative Fields

In Section 18.1 we learned that if $\mathbf{F} = P\mathbf{i}+Q\mathbf{j}$ is defined on a simply connected region in the xy-plane and if $\mathbf{curl}\,\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is conservative. Now that we have Stokes' theorem, this result can be extended to a field $\mathbf{F} = P\mathbf{i}+Q\mathbf{j}+R\mathbf{k}$ defined on a simply connected region in space.

Theorem 18.6.2. *Let* $\mathbf{F}$ *be defined on a simply connected region in space. If* $\mathbf{curl}\,\mathbf{F} = \mathbf{0}$*, then* $\mathbf{F}$ *is conservative.*

Proof

We provide only a sketch of the proof of this result. Let C be a simple closed curve situated in the simply connected region. To avoid topological complexities, we assume that it bounds an orientable surface $\mathcal{S}$. To show that $\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$, we use the same short argument as in Section 18.2:

$$\oint\limits_{C} \mathbf{F}\cdot\, d\mathbf{r} = \int\limits_{\mathcal{S}} (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS = \int\limits_{\mathcal{S}} 0\,dS = 0.$$

•

It follows from Theorem 18.6.2 that every central field $\mathbf{F}$ is conservative because a straightforward calculation shows that the curl of a central field is $\mathbf{0}$. (See Exercises 6 and 7 in Section 18.4.) Moreover, $\mathbf{F}$ is defined either throughout space or everywhere except at the center of the field.

Exercise 26 of Section 18.4 presents a purely geometric argument for why a central field is conservative.

In Sections 18.7 and 18.9 we will show how Stokes's theorem is applied in the theory of electromagnetism.

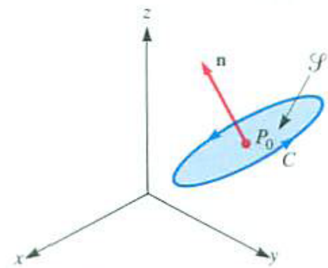


Figure 18.6.9:

Why Curl is Called Curl

Let $\mathbf{F}$ be a vector field describing the flow of a fluid, as in Section 18.1. Stokes's theorem will give a physical interpretation of $\mathbf{curl\,F}$.

Consider a fixed point P_0 in space. Imagine a *small* circular disk $\mathcal{S}$ with center P_0. Let C be the boundary of $\mathcal{S}$ oriented in such a way that C and $\mathbf{n}$ fit the right-hand rule. (See Figure 18.6.9)

Now examine the two sides of the equation

$$\int_{\mathcal{S}} (\mathbf{curl\,F}) \cdot \mathbf{n}\; dS = \oint_C \mathbf{F} \cdot \mathbf{T}\; ds. \tag{18.6.7}$$

The right side of (18.6.7) measures the tendency of the fluid to move along C (rather than, say, perpendicular to it.) Thus $\oint_C \mathbf{F} \cdot \mathbf{T}\; ds$ might be thought of as the "circulation" or "whirling tendency" of the fluid along C. For each tilt of the small disk $\mathcal{S}$ at P_0 — or, equivalently, each choice of unit normal vector $\mathbf{n}$ – the line integral $\oint_C \mathbf{F} \cdot \mathbf{T}\; ds$ measures a corresponding circulation. It records the tendency of a paddle wheel at P_0 with axis along $\mathbf{n}$ to rotate. (See Figure 18.6.10.)

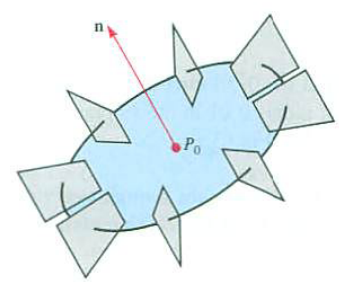


Figure 18.6.10:

Consider the left side of (18.6.7). If $\mathcal{S}$ is small, the integrand is almost constant and the integral is approximately

$$(\mathbf{curl\,F})_{P_0} \cdot \mathbf{n} \cdot \text{Area of } \mathcal{S}, \tag{18.6.8}$$

where $(\mathbf{curl\,F})_{P_0}$ denotes the curl of $\mathbf{F}$ evaluated at P_0.

Keeping the center of $\mathcal{S}$ at P_0, vary the vector $\mathbf{n}$ by tilting the disk $\mathcal{S}$. For which choice of $\mathbf{n}$ will (18.6.8) be largest? Answer: For that $\mathbf{n}$ which has the same direction as the fixed vector $(\mathbf{curl\,F})_{P_0}$. With that choice of $\mathbf{n}$, (18.6.8) becomes

$$\|(\mathbf{curl\,F})_{P_0}\| \text{ Area of } \mathcal{S}\ .$$

The physical interpretation of curl

Thus a paddle wheel placed in the fluid at P_0 *rotates most quickly when its axis is in the direction of* $\mathbf{curl\,F}$ at P_0. The *magnitude* of $\mathbf{curl\,F}$ is a measure of *how fast* the paddle wheel can rotate when placed at P_0. Thus $\mathbf{curl\,F}$ records the direction and magnitude of maximum circulation at a given point. If $\mathbf{curl\,F}$ is $\mathbf{0}$, there is no tendency of the fluid to rotate; that is why such vector fields are called irrotational.

A Vector Definition of Curl

In Section 18.1 $\mathbf{curl}\,\mathbf{F}$ was defined in terms of the partial derivatives of the components of $\mathbf{F}$. By Stokes' theorem, $\mathbf{curl}\,\mathbf{F}$ is related to the circulation, $\oint_C \mathbf{F}\cdot\, d\mathbf{r}$. We exploit this relation to obtain a new view of $\mathbf{curl}\,\mathbf{F}$, free of coordinates.

Let P_0 be a point in space and let $\mathbf{n}$ be a unit vector. Consider a small disk $\mathcal{S}_{\mathbf{n}}(a)$, perpendicular to $\mathbf{n}$, whose center is P_0, and which has a radius a. Let $C_{\mathbf{n}}(a)$ be the boundary of $\mathcal{S}_{\mathbf{n}}(a)$, oriented to be compatible with the right-hand rule. Then

$$\int\limits_{\mathcal{S}_{\mathbf{n}}(a)} (\mathbf{curl}\,\mathbf{F})\cdot\mathbf{n}\; dS = \oint\limits_{C_{\mathbf{n}}(a)} \mathbf{F}\cdot\, d\mathbf{r}.$$

As in our discussion of the physical meaning of curl, we see that

$$(\mathbf{curl}\,\mathbf{F})(P_0)\cdot\mathbf{n}\cdot\text{Area of }\mathcal{S}_{\mathbf{n}}(a) \approx \oint\limits_{C_{\mathbf{n}}(a)} \mathbf{F}\cdot d\mathbf{r},$$

or

$$(\mathbf{curl}\,\mathbf{F})(P_0)\cdot\mathbf{n} \approx \frac{\oint_{C_{\mathbf{n}}(a)} \mathbf{F}\cdot\, d\mathbf{r}}{\text{Area of }\mathcal{S}_{\mathbf{n}}(a)}.$$

Thus

$$(\mathbf{curl}\,\mathbf{F})(P_0)\cdot\mathbf{n} = \lim_{a\to 0}\frac{\oint_{C_{\mathbf{n}}(a)} \mathbf{F}\cdot\, d\mathbf{r}}{\text{Area of }\mathcal{S}_{\mathbf{n}}(a)}. \tag{18.6.9}$$

Equation (18.6.9) gives meaning to the component of $(\mathbf{curl}\,\mathbf{F})(P_0)$ in any direction $\mathbf{n}$. So the magnitude and direction of $\mathbf{curl}\,\mathbf{F}$ at P_0 can be described in terms of $\mathbf{F}$, without looking at the components of $\mathbf{F}$.

The magnitude of $(\mathbf{curl}\,\mathbf{F})_{P_0}$ is the maximum value of

$$\lim_{a\to 0}\frac{\oint_{C_{\mathbf{n}}(a)} \mathbf{F}\cdot\, d\mathbf{r}}{\text{Area of }\mathcal{S}_{\mathbf{n}}(a)}, \tag{18.6.10}$$

for all unit vectors $\mathbf{n}$.
The direction of $(\mathbf{curl}\,\mathbf{F})_{P_0}$ is given by the vector $\mathbf{n}$ that maximizes the limit (18.6.10).

EXAMPLE 3 Let $\mathbf{F}$ be a vector field such that at the origin $\mathbf{curl}\,\mathbf{F} = 2\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$. Estimate $\oint_C \mathbf{F}\cdot\, d\mathbf{r}$ if C encloses a disk of radius 0.01 in the

xy-plane with center $(0,0,0)$. C is swept out clockwise. (See Figure 18.6.11.)

SOLUTION Let $\mathcal{S}$ be the disk whose border is C. Choose the normal to $\mathcal{S}$ that is consistent with the orientation of C and the right-hand rule. That choice is $-\mathbf{k}$. Thus

$$(\mathbf{curl}\,\mathbf{F})\cdot(-\mathbf{k}) \approx \frac{\oint_C \mathbf{F}\cdot\ d\mathbf{r}}{\text{Area of } \mathcal{S}}.$$

The area of $\mathcal{S}$ is $\pi(0.01)^2$ and $\mathbf{curl}\,\mathbf{F} = 2\mathbf{i}+3\mathbf{j}+4\mathbf{k}$. Thus

$$(2\mathbf{i}+3\mathbf{j}+4\mathbf{k})\cdot(-\mathbf{k}) \approx \frac{\oint_C \mathbf{F}\cdot\ d\mathbf{r}}{\pi(0.01)^2}.$$

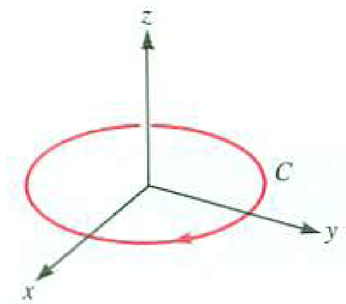


Figure 18.6.11:

From this it follows that

$$\oint\limits_C \mathbf{F}\cdot\ d\mathbf{r} \approx -4\pi(0.01)^2.$$

◇

In a letter to the mathematician Tait written on November 7, 1870, Maxwell offered some names for $\nabla\times\mathbf{F}$:

> Here are some rough-hewn names. Will you like a good Divinity shape their ends properly so as to make them stick? ...
> The vector part $\nabla\times\mathbf{F}$ I would call the twist of the vector function. Here the word twist has nothing to do with a screw or helix. The word *turn* ... would be better than twist, for twist suggests a screw. Twirl is free from the screw motion and is sufficiently racy. Perhaps it is too dynamical for pure mathematicians, so for Cayley's sake I might say Curl (after the fashion of Scroll.)

His last suggestion, "curl," has stuck.

Proof of Stokes' Theorem

We include this proof because it reviews several basic ideas. The proof uses Green's theorem, the normal to a surface $z=f(x,y)$, and expressing an integral over a surface as an integral over its shadow on a plane. The approach is straightforward. As usual, we begin by expressing the theorem in terms of components. We will assume that the surface $\mathcal{S}$ meets each line parallel to an axis in at most one point. That permits us to project $\mathcal{S}$ onto each coordinate plane in an one-to-one fashion.

To begin we write $\mathbf{F}(x, y, z)$ as $P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$, or, simply $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$. We will project $\mathcal{S}$ onto the xy-plane, so write the equation for $\mathcal{S}$ as $z - f(x, y) = 0$. A unit normal to $\mathcal{S}$ is

$$\mathbf{n} = \frac{-\frac{\partial f}{\partial x}\mathbf{i} - \frac{\partial f}{\partial y}\mathbf{j} + \mathbf{k}}{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}}.$$

(Since the $\mathbf{k}$ component of $\mathbf{n}$ is positive, it is the correct normal, given by the right-hand rule.) Let C^* be the projection of C on the xy-plane, swept out counterclockwise.

See Exercise 9.

A straightforward computation shows that Stokes' theorem, expressed in components, reads

$$\begin{aligned}&\int\limits_C P\ dx + Q\ dy + R\ dz \\ &\quad = \int\limits_{\mathcal{S}} \frac{\left(\frac{\partial R}{\partial x} - \frac{\partial Q}{\partial z}\right)\left(-\frac{\partial f}{\partial x}\right) - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\left(-\frac{\partial f}{\partial y}\right) + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)(1)}{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}}\ dS.\end{aligned}$$

As expected, this equation reduces to three equations, one for P, one for Q, and one for R.

We will establish the result for P, namely

$$\int\limits_C P\ dx = \int\limits_{\mathcal{S}} \frac{\frac{\partial P}{\partial z}\left(-\frac{\partial f}{\partial y}\right) - \frac{\partial P}{\partial y}(1)}{\sqrt{(\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2 + 1}}\ dS. \tag{18.6.11}$$

To change the integral over $\mathcal{S}$ to an integral over its projection, $\mathcal{S}^*$, on the xy-plane, we replace dS by $\sqrt{(\partial f/\partial x)^2 + (\partial f/\partial y)^2 + 1}\ dA$. At the same time we project C onto a counterclockwise curve C^*. The square roots cancel leaving us with this equation in the xy-plane:

$$\int\limits_{C*} P(x, y, f(x, y))\ dx = \int\limits_R \left(-\frac{\partial P}{\partial z}\frac{\partial f}{\partial y} - \frac{\partial P}{\partial y}\right)\ dA. \tag{18.6.12}$$

Finally, we apply Green's theorem to the left side of (18.6.12), and obtain:

$$\int\limits_{C^*} P(x, y, f(x, y))\ dx = \int\limits_{\mathcal{S}^*} -\frac{\partial P(x, y, f(x, y))}{\partial y}\ dA.$$

Be sure you understand each of the four steps in this proof, and why they are valid.

But
$$\frac{\partial P(x,y,f(x,y))}{\partial y} = \frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}\frac{\partial f}{\partial y}. \tag{18.6.13}$$
Combining (18.6.12) and (18.6.13) completes the proof of (18.6.11).

In this proof we assumed that the surface $\mathcal{S}$ has a special form, meeting lines parallel to an axis just once. However, more general surfaces, such as the surface of a sphere or a polyhedron can be cut into pieces of the type treated in the proof. Exercise 48 shows why this observation then implies that Stokes' Theorem holds in these cases also.

Summary

Stokes' Theorem relates the circulation of a vector field over a closed curve C to the integral over a surface $\mathcal{S}$ that C bounds. The integrand over the surface is the component of the curl of the field perpendicular to the surface,
$$\int\limits_C \mathbf{F}\cdot\, d\mathbf{r} = \int\limits_{\mathcal{S}} (\mathbf{curl}\,\mathbf{F})\cdot\mathbf{n}\; dS.$$
The normal $\mathbf{n}$ is the normal vector to $\mathcal{S}$ given by the right-hand rule.

EXERCISES for Section 18.6 *Key:* R–routine, M–moderate, C–challenging

1.[C] We dealt only with the component P. What is the analog of (18.6.11) for Q? Prove it. HINT: The steps would parallel the steps used for P.

2.[R] State Stokes' Theorem (symbols permitted).

3.[R] State Stokes' Theorem in words (symbols not permitted).

4.[M] Explain why (18.6.5) holds if $\mathcal{S}_1$ and $\mathcal{S}_2$ together form the boundary surface $\mathcal{S}$ of a solid region R. Use the Divergence Theorem, not Stokes' Theorem.

5.[R] Let $F(r)$ be an antiderivative of $f(r)$. Show that $f(r)\widehat{\mathbf{r}}$ is the gradient of $F(r)$, hence is conservative. NOTE: $f(r)\frac{\mathbf{r}}{r} = f(r)\widehat{\mathbf{r}}$.

6.[M] Show that a central field $f(r)\widehat{\mathbf{r}}$ is conservative by showing that it is irrotational and defined on a simply connected region. HINT: Express $\widehat{\mathbf{r}}$ in terms of x, y and z. NOTE: See also Exercise 47.

7.[R]

(a) Use the fact that a gradient, ∇f, is conservative, to show that its curl is $\mathbf{0}$.

(b) Compute $\nabla\times\nabla f$ in terms of components to show that the curl of a gradient is $\mathbf{0}$.

8.[C] (See also Exercises 5 and 6.)

Sam: The only conservative fields in space that I know are the "inverse square central fields" with centers anywhere I please.

Jane: There are a lot more.

Sam: Oh?

Jane: Just start with any scalar function $f(x, y, z)$ with continuous partial derivation of the first and second orders. Then its gradient will be a conservative field.

Sam: O.K. But I bet there are still more.

Jane: No. I got them all.

Question: Who is right?

Exercises 9 to 14 concern the proof of Stokes' Theorem.

9.[C] Carry out the calculations in the proof that translated Stokes' Theorem into an equation involving the components P, Q, and R.

10.[C] Draw a picture of $\mathcal{S}$, $\mathcal{S}^*$, C and C^* that appear in the proof of Stokes' Theorem.

11.[C] Write the four steps involved in the proof of Stokes' Theorem, giving an explanation for each step.

12.[C] In the proof of Stokes' Theorem we used a normal $\mathbf{n}$. Show that it is the "correct" one, compatible with counterclockwise orientation of C^*.

13.[C]

(a) State Stokes' Theorem for $\int_C Q\,dy$.

(b) Prove Stokes' Theorem for $\int_C Q\,dy$.

(c) State Stokes' Theorem for $\int_C R\,dz$.

(d) Prove Stokes' Theorem for $\int_C R\,dz$.

14.[C] Draw a picture of $\mathcal{S}$, $\mathcal{S}^*$, C and C^* that appear in the proof.

Exercises 15 to 17 prepare you for Exercise 18.

15.[M] Assume that $\mathbf{G}$ is the curl of another vector field $\mathbf{F}$, $\mathbf{G} = \nabla\times\mathbf{F}$. Let $\mathcal{S}$ be a surface that bounds a solid region V. Let C be a closed curve on the surface $\mathcal{S}$ breaking $\mathcal{S}$ into two pieces $\mathcal{S}_1$ and $\mathcal{S}_2$.

16.[M] Using the Divergence Theorem, show that $\int_{\mathcal{S}} \mathbf{G}\cdot\mathbf{n}\,dS = 0$.

17.[M] Using Stokes' Theorem, show that $\int_{\mathcal{S}} \mathbf{G}\cdot\mathbf{n}\,dS = 0$. HINT: Break the integral into integrals over $\mathcal{S}_1$ and $\mathcal{S}_2$.

18.[R] Let $\mathbf{F} = e^{xy}\mathbf{i} + \tan(3yz)\mathbf{j} + 5z\mathbf{k}$ and $\mathcal{S}$ be the tetrahedron whose vertices are $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. Let $\mathcal{S}_1$ be the base of $\mathcal{S}$ in the xy-plane and $\mathcal{S}_2$ consist of the other three faces. Find $\int_{\mathcal{S}} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$. HINT: think about the preceding two exercises.

19.[R] Assume that $\mathbf{F}$ is defined everywhere except on the z-axis and is irrotational. The curves C_1, C_2, C_3, and C_4 are as shown in Figure 18.6.12. What, if anything, can be said about

$$\oint_{C_1} \mathbf{F} \cdot d\mathbf{r}, \quad \oint_{C_2} \mathbf{F} \cdot d\mathbf{r}, \quad \oint_{C_3} \mathbf{F} \cdot d\mathbf{r}, \quad \text{and} \quad \oint_{C_4} \mathbf{F} \cdot d\mathbf{r}$$

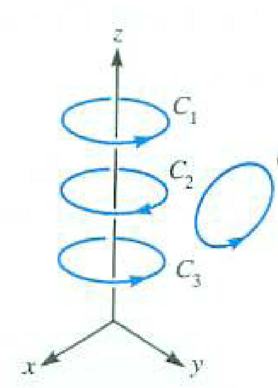


Figure 18.6.12:

In Exercises 20 to 23 verify Stokes' Theorem for the given $\mathbf{F}$ and surface $\mathcal{S}$.

20.[R] $\mathbf{F} = xy^2\mathbf{i} + y^3\mathbf{j} + y^2z\mathbf{k}$; $\mathcal{S}$ is the top half of the sphere $x^2 + y^2 + z^2 = 1$.

21.[R] $\mathbf{F} = y\mathbf{i} + xz\mathbf{j} + x^2\mathbf{k}$; $\mathcal{S}$ is the triangle with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.

22.[R] $\mathbf{F} = y^5\mathbf{i} + x^3\mathbf{j} + z^4\mathbf{k}$; $\mathcal{S}$ is the portion of $z = x^2 + y^2$ below the plane $z = 1$.

23.[R] $\mathbf{F} = -y\mathbf{i} + x\mathbf{j} + z\mathbf{k}$, $\mathcal{S}$ is the portion of the cylinder $z = x^2$ inside the cylinder $x^2 + y^2 = 4$.

24.[R] Evaluate as simply as possible $\int_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n}\, dS$, where $\mathbf{F}(x,y,z) = x\mathbf{i} - y\mathbf{j}$ and $\mathcal{S}$ is the surface of the cube bounded by the three coordinate planes and the planes $x = 1$, $y = 1$, $z = 1$, exclusive of the surface in the plane $x = 1$. (Let $\mathbf{n}$ be outward from the cube.)

25.[R] Using Stokes' Theorem, evaluate $\int_{\mathcal{S}} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$, where $\mathbf{F} = (x^2 + y - 4)\mathbf{i} + 3xy\mathbf{j} + (2xz + z^2)\mathbf{k}$, and $\mathcal{S}$ is the portion of the surface $z = 4 - (x^2 + y^2)$ above the xy plane. (Let $\mathbf{n}$ be the upward normal.)

In each of Exercises 26 to 29 use Stokes' Theorem to evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ for the given $\mathbf{F}$ and C. In each case assume that C is oriented counterclockwise when viewed from above.

26.[R] $\mathbf{F} = \sin(xy)\mathbf{i}$; C is the intersection of the plane $x + y + z = 1$ and the cylinder $x^2 + y^2 = 1$.

27.[R] $\mathbf{F} = e^x\mathbf{j}$; C is the triangle with vertices $(2,0,0)$, $(0,3,0)$ and $(0,0,4)$.

28.[R] $\mathbf{F} = xy\mathbf{k}$; C is the intersection of the plane $z = y$ with the cylinder $x^2 - 2x + y^2 = 0$.

29.[R] $\mathbf{F} = \cos(x + z)\mathbf{j}$; C is the boundary of the rectangle with vertices $(1,0,0)$, $(1,1,1)$, $(0,1,1)$, and $(0,0,0)$.

30.[R] Let $\mathcal{S}_1$ be the top half and $\mathcal{S}_2$ the bottom half of a sphere of radius a in space. Let $\mathbf{F}$ be a vector field defined on the sphere and let $\mathbf{n}$ denote an exterior normal to the sphere. What relation, if any, is there between $\int_{\mathcal{S}_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$ and $\int_{\mathcal{S}_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$?

31.[R] Let $\mathbf{F}$ be a vector field throughout space such that $\mathbf{F}(P)$ is perpendicular to the curve C at each point P on C, the boundary of a surface $\mathcal{S}$. What can one conclude about

$$\int_{\mathcal{S}} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS?$$

32.[R] Let C_1 and C_2 be two closed curves in the

xy-plane that encircle the origin and are similarly oriented, as in Figure 18.6.13.

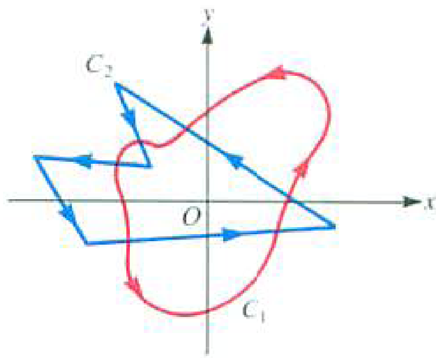


Figure 18.6.13:

Let **F** be a vector field defined throughout the plane except at the origin. Assume that $\nabla \times \mathbf{F} = \mathbf{0}$.

(a) Must $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$?

(b) What, it any, relation exists between $\oint_{C_1} \mathbf{F} \cdot d\mathbf{r}$ and $\oint_{C_2} \mathbf{F} \cdot d\mathbf{r}$?

33.[R] Let **F** be defined everywhere in space except on the z-axis. Assume also that **F** is irrotational, $\oint_{C_1} \mathbf{F} \cdot d\mathbf{r} = 3$, and $\oint_{C_2} \mathbf{F} \cdot d\mathbf{r} = 5$. (See Figure 18.6.14.) What if, anything, can be said about

(a) $\oint_{C_3} \mathbf{F} \cdot d\mathbf{r}$,

(b) $\oint_{C_4} \mathbf{F} \cdot d\mathbf{r}$?

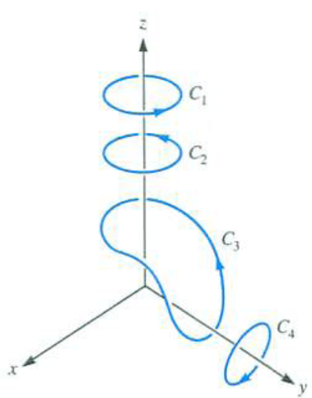


Figure 18.6.14:

34.[R] Which of the following sets are connected? simply connected?

(a) A circle ($x^2 + y^2 = 1$) in the xy-plane

(b) A disk ($x^2 + y^2 \leq 1$) in the xy-plane

(c) The xy-plane from which a circle is removed

(d) The xy-plane from which a disk is removed

(e) The xy-plane from which one point is removed

(f) xyz-space from which one point is removed

(g) xyz-space from which a sphere is removed

(h) xyz-space from which a ball is removed

(i) A solid torus (doughnut)

(j) xyz-space from which a solid torus is removed

(k) A coffee cup with one handle

(l) xyz-space from which a solid doughnut is removed

35.[R] Which central fields have curl **0**?

36.[R] Let $\mathcal{V}$ be the solid bounded by $z = x + 2$, $x^2 + y^2 = 1$, and $z = 0$. Let $\mathcal{S}_1$ be the portion of the plane $z = x + 2$ that lies within the cylinder $x^2 + y^2 = 1$. Let C be the boundary of $\mathcal{S}_1$, with a counterclockwise orientation (as viewed from above). Let $\mathbf{F} = y\mathbf{i} + xz\mathbf{j} + (x + 2y)\mathbf{k}$. Use Stokes' Theorem for $\mathcal{S}_1$ to evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$.

37.[R] (See Exercise 36.) Let $\mathcal{S}_2$ be the curved surface of $\mathcal{V}$ together with the base of $\mathcal{V}$. Use Stokes' Theorem for $\mathcal{S}_2$ to evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$.

38.[R] Verify Stokes' theorem for the special case when **F** has the form ∇f, that is, is a gradient field.

39.[R] Let **F** be a vector field defined on the surface $\mathcal{S}$ of a convex solid. Show that $\int_{\mathcal{S}} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = 0$

(a) by the Divergence Theorem,

(b) by drawing a closed curve on C on $\mathcal{S}$ and using Stokes' Theorem on the two parts into which C divides $\mathcal{S}$.

40.[R] Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ as simply as possible if $\mathbf{F}(x,y,z) = (-y\mathbf{i} + x\mathbf{j})/(x^2+y^2)$ and C is the intersection of the plane $z = 2x + 2y$ and the paraboloid $z = 2x^2+3y^2$ oriented counterclockwise as viewed from above.

41.[R] Let $\mathbf{F}(x,y)$ be a vector field defined everywhere in the plane except at the origin. Assume that $\nabla \times \mathbf{F} = \mathbf{0}$. Let C_1 be the circle $x^2+y^2 = 1$ counterclockwise; let C_2 be the circle $x^2+y^2 = 4$ clockwise; let C_3 be the circle $(x-2)^2+y^2 = 1$ counterclockwise; let C_4 be the circle$(x-1)^2+y^2 = 9$ clockwise. Assuming that $\oint_{C_1} \mathbf{F} \cdot d\mathbf{r}$ is 5, evaluate

(a) $\oint_{C_2} \mathbf{F} \cdot d\mathbf{r}$

(b) $\oint_{C_3} \mathbf{F} \cdot d\mathbf{r}$

(c) $\oint_{C_4} \mathbf{F} \cdot d\mathbf{r}$.

42.[M] Let $\mathbf{F}(x,y,z) = \mathbf{r}/\|\mathbf{r}\|^a$, where $\mathbf{r} = \mathbf{i} + u\mathbf{j} + z\mathbf{k}$ and a is a fixed real number.

(a) Show that $\nabla \times \mathbf{F} = \mathbf{0}$.

(b) Show that $\mathbf{F}$ is conservative.

(c) Exhibit a scalar function f such that $\mathbf{F} = \nabla f$.

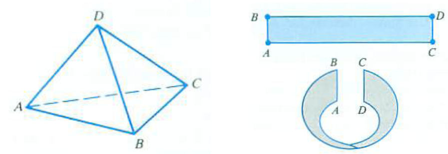
43.[M] Let $\mathbf{F}$ be defined throughout space and have continuous divergence and curl.

(a) For which $\mathbf{F}$ is $\int_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n}\, dS = 0$ for all spheres $\mathcal{S}$?

(b) For which $\mathbf{F}$ is $\oint_C \mathbf{F} \cdot\, d\mathbf{r} = 0$ for all circles C?

44.[M] Let C be the curve formed by the intersection of the plane $z = x$ and the paraboloid $z = x^2 + y^2$. Orient C to be counterclockwise when viewed from above. Evaluate $\oint_C (xyz\, dx + x^2\, dy + xz\, dz)$.

45.[M] Assume that Stokes' Theorem is true for triangles. Deduce that it then holds for the surface $\mathcal{S}$ in Figure 18.6.15(a), consisting of the three triangles DAB, DBC, DCA, and the curve $ABCA$.



(a) (b)

Figure 18.6.15:

46.[C] A Möbius band can be made by making a half-twist in a narrow rectangular strip, bringing the two ends together, and fastening them with glue or tape. See Figure 18.6.15(b).

(a) Make a Möbius band.

(b) Letting a pencil represent a normal $\mathbf{n}$ to the band, check that the Möbius band is not orientable.

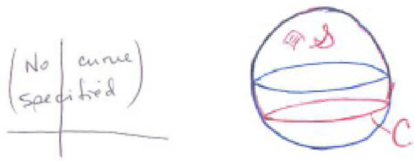
(c) If you form a band by first putting in a full twist (360°), is it orientable?

(d) What happens when you cut the bands in (a) and (c) down the middle? one third of the way from one edge to the other?

47.[C]

(a) Explain why the line integral of a central vector field $f(r)\widehat{\mathbf{r}}$ around the path in Figure 18.6.16(a) is 0.

(b) Deduce from (a) and the coordinate-free view of curl that the curl of a central field is $\mathbf{0}$.



(a) (b)

Figure 18.6.16:

48.[C]

(a) The proof of Stokes' Theorem we gave would not apply to surfaces that are more complicated, such as the "top three fourths of a sphere," as shown in Figure 18.6.16(b). However, how could you cut $\mathcal{S}$ into pieces to each of which the proof applies? (Describe them in general terms, in words.)

(b) How could you use (a) to show that Stokes' Theorem holds for C and $\mathcal{S}$ in Figure 18.6.16(b)

49.[M] Sam has a different way to
n.

Sam: I think the book's way of ch
plicated.

Jane: OK. How would you do it?

Sam: Glad you asked. First, I w
normal **n** at one point on th

Jane: That's a good start.

Sam: Then I choose unit normals
everywhere on the surface st
choice.

Jane: And how would you finish?

Sam: My last step is to orient the
be compatible with the right

Would this proposal work? If it d
with the approach in the text.

18.7 Connections Between the Electric Field and $\widehat{\mathbf{r}}/\|\mathbf{r}\|^2$

Even if you are not an engineer or physicist, as someone living in the 21st century you are surrounded by devices that depend on electricity. For that reason we now introduce one of the four equations that explain all of the phenomena of electricity and magnetism. Later in the chapter we will turn to the other three equations, all of which are expressed in terms of vector fields. The chapter concludes with a detailed description of how James Clerk Maxwell, using just these four equations, predicted that light is an electromagnetic phenomenon. Our explanation does not assume any prior knowledge of physics.

The Electric Field Due To a Single Charge

The starting point is some assumptions about the fundamental electrical charges, electrons and protons. An electron has a negative charge and a proton has a positive charge of equal absolute value. Two like charges exert a force of repulsion on each other; unlike charges attract each other.

Let C and P denote the location of charges q and q_0, respectively. Let $\mathbf{r}$ be the vector from C to P, as in Figure 18.7.1, so $r = \|\mathbf{r}\|$ is the distance between the two charges.

If both q and q_0 are protons or both are electrons, the force pushes the charges further apart. If one is a proton and the other is an electron, the force draws them closer. In both cases the magnitude of the force is inversely proportional to r^2, the square of the distance between the charges.

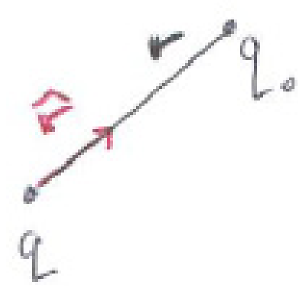


Figure 18.7.1: ARTIST: Please modify labeling to reflect that the charges are located at C and P with charges q and q_0, respectively.

Assume that q is positive, that is, is the charge of a proton. The magnitude of the force it exerts on charge q_0 is proportional to q and also proportional to q_0. It is also inversely proportional to r^2. So, for some constant k, the magnitude of the force is of the form

$$k\frac{q\,q_0}{r^2}.$$

It is directed along the vector $\mathbf{r}$. If q_0 is also positive, it is in the same direction as $\mathbf{r}$. If q_0 is negative, it is in the direction of $-\mathbf{r}$. We can summarize these observations in one vector equation

$$\mathbf{F} = k\frac{q\,q_0}{r^2}\widehat{\mathbf{r}} \tag{18.7.1}$$

where the constant k is positive.

Read ϵ_0 as "epsilon zero" or "epsilon null."

For convenience in later calculations, k is replaced by $1/(4\pi\epsilon_0)$ The value of ϵ_0 depends on the units in which charge, distance, and force are measured. Then (18.7.1) is written

$$\mathbf{F} = \frac{q\,q_0}{4\pi\epsilon_0 r^2}\widehat{\mathbf{r}}.$$

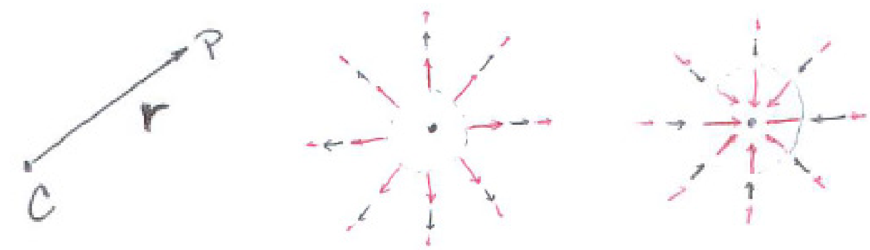
Physicists associate with a charge q a vector field. This field in turn exerts a force on other charges.

Consider a positive charge q at point C.

It "creates" a central inverse-square vector field $\mathbf{E}$ with center at C. It is defined everywhere except at C. Its value at a typical point P is

$$\mathbf{E}(P) = \frac{q\,\widehat{\mathbf{r}}}{4\pi\epsilon_0 r^2}$$

where $\overrightarrow{\mathbf{r}} = \overrightarrow{CP}$, as in Figure 18.7.2.



(a) (b) (c)

Figure 18.7.2:

The value of $\mathbf{E}$ depends only on q and the vector from C to P.

To find the force exerted by charge q on charge q_0 at P just multiply $\mathbf{E}$ by q_0, obtaining

$$\mathbf{F} = q_0\mathbf{E} \tag{18.7.2}$$

The field $\mathbf{E}$, which is a sheer invention, can be calculated in principle by putting a charge q_0 at P, observing the force $\mathbf{F}$ and then dividing $\mathbf{F}$ by q_0. The field $\mathbf{E}$ enables the charge q to "act at a distance" on other charges. It plays the role of a rubber band or a spring.

The Electric Field Due to a Distribution of Charge

Electrons and protons usually do not live in isolation. Instead, charge may be distributed on a line, a curve, a surface or in space.

Imagine a total charge Q occupying a region R in space. The density of the charge varies from point to point. Denote the density at P by $\delta(P)$. Like the density of mass it is defined as a limit as follows. Let $V(r)$ be a small ball of radius r and center at P. Then we have the definition

$$\delta(P) = \lim_{r\to 0^+} \frac{\text{charge in } V(r)}{\text{volume of } V(r)}.$$

The charge in $V(r)$ is approximately the volume of $V(r)$ times $\delta(P)$. We will be interested only in uniform charges, where the density is constant, with the fixed value δ. Thus the charge in a region of volume V is δV.

The field due to a uniform charge Q distributed in a region R is the sum of the fields due to the individual point charges in Q.

To describe that field we need the concept of the integral of a vector field. The definition is similar to the definition of the definite integral in Section 6.2. Let $\mathbf{F}(P)$ be a continuous vector field defined on some solid region R. Break R into regions $R_1, R_2, \ldots, R_n$ and choose a point P_i in R_i, $1 \leq i \leq n$. Let the volume of R_i be V_i. The sums $\sum_{i=1}^{n} \mathbf{F}(P_i)V_i$ have a limit as all R_i are chosen smaller and smaller. This limit, denoted $\int_R \mathbf{F}(P)\ dV$ is called the **integral of F over** R. Computationally, this integral can be computed componentwise. For example, if $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + P_3\mathbf{k}$ then $_R\mathbf{F}(P)$ then $\int_R \mathbf{F}\ dV = \int_R F_1\ dV\mathbf{i} + \int_R F_1\ dV\mathbf{j} + \int_R F_1\ dV\mathbf{k}$. Similar definitions hold for vector fields defined on surfaces or curves.

To estimate this field we partition R into small regions $R_1, R_2, \ldots R_n$ and choose a point P_i in R_i, $i = 1, 2 \ldots, n$. The volume of R_i is V_i. The charge in R_i is δV_i, where δ is the density of the charge. Figure 18.7.3 shows this contribution to the field at a point P.

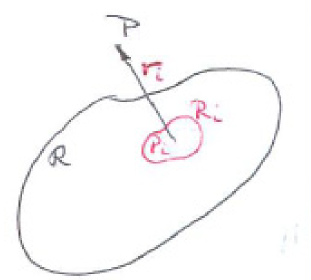


Figure 18.7.3:

Let $\mathbf{r}_i$ be the vector from P_i to P, and $r_i = |\mathbf{r}_i|$. Then the field due to the charge in this small patch R_i is approximately

$$\frac{\delta\ \widehat{\mathbf{r}}_i\ V_i}{4\pi\epsilon_0\ r_i^2}.$$

As an estimate of the field due to Q, we have the sum

$$\sum_{i=1}^{n} \frac{\delta\ \widehat{\mathbf{r}}_i\ V_i}{4\pi\epsilon_0 r_i^2}.$$

Taking limits as all the regions R_i are chosen smaller, we have

$$\mathbf{E}(P) = \text{Field at } P = \int\limits_R \frac{\delta\ \widehat{\mathbf{r}}}{4\pi\epsilon_0 r^2}\ dV$$

Factoring out the constant $\delta/4\pi\epsilon_0$, we have

$$\mathbf{E}(P) = \frac{\delta}{4\pi\epsilon_0} \int\limits_R \frac{\widehat{\mathbf{r}}}{r^2}\ dV$$

That is an integral over a solid region. If the charge is just on a surface S with uniform surface density σ, the field would be given by

$$\mathbf{E}(P) = \frac{\sigma}{4\pi\epsilon_0} \int\limits_S \frac{\widehat{\mathbf{r}}}{r^2}\ dS.$$

If the charge lies on a line or a curve C, with uniform density λ, then

$$\mathbf{E}(P) = \frac{\lambda}{4\pi\epsilon_0} \int_C \frac{\widehat{\mathbf{r}}}{r^2}\, ds.$$

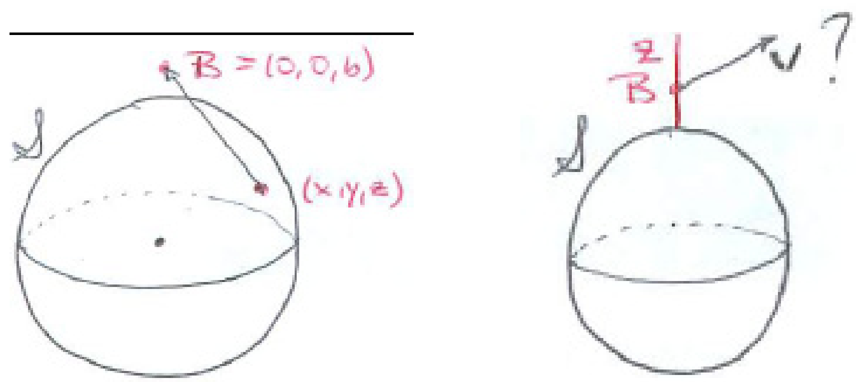
To illustrate the definition we compute one such field value directly. In Example 2 we solve the same problem much more simply.

EXAMPLE 1 A charge Q is uniformly distributed on a sphere of radius a, $\mathcal{S}$. Find the electrostatic field $\mathbf{E}$ at a point B a distance $b > a$ from the center of the sphere.
SOLUTION We evaluate

$$\frac{\sigma}{4\pi\epsilon_0} \int_S \frac{\widehat{\mathbf{r}}}{r^2}\, dS. \tag{18.7.3}$$

Note that $\sigma = Q/4\pi a^2$, since the charge is uniform over an area of $4\pi a^2$.

Place a rectangular coordinate system with its origin at the center of the sphere and the z-axis on B, so that $B = (0, 0, b)$, as in Figure 18.7.4(a). Before we start to evaluate an integral, let us use the symmetry of the sphere



(a) (b)

Figure 18.7.4:

to predict something about the vector $\mathbf{E}(B)$. Could it look like the vector $\mathbf{v}$, which is not parallel to the z-axis, as in Figure 18.7.4(b)?

If you spin the sphere around the z-axis, the vector $\mathbf{v}$ would change. But the sphere is unchanged and so is the charge. So $\mathbf{E}(B)$ must be parallel to the z-axis. That means we know its x- and y-components are both 0. So we must find just its z-component, which is $\mathbf{E}(B) \cdot \mathbf{k}$.

Let (x, y, z) be a typical point on the sphere $\mathcal{S}$. Then

$$\mathbf{r} = (0\mathbf{i} + 0\mathbf{j} + b\mathbf{k}) - (x\mathbf{i} + y\mathbf{j} - z\mathbf{k}) = -x\mathbf{i} - y\mathbf{j} + (b - z)\mathbf{k}. \tag{18.7.4}$$

So

$$\frac{\widehat{\mathbf{r}}}{r^2} = \frac{\mathbf{r}}{r^3} = \frac{-x\mathbf{i} - y\mathbf{j} + (b - z)\mathbf{k}}{(\sqrt{x^2 + y^2 + b^2 - 2bz + z^2})^3} = \frac{-x\mathbf{i} - y\mathbf{j} + (b - z)\mathbf{k}}{(a^2 + b^2 - 2bz)^{3/2}}. \tag{18.7.5}$$

We need only the z-component of this,

$$\frac{b - z}{(a^2 + b^2 - 2bz)^{3/2}}.$$

The magnitude of $\mathbf{E}(B)$ is therefore

$$\frac{\sigma}{4\pi\epsilon_0} \int\limits_S \frac{b - z}{(a^2 + b^2 - 2bz)^{3/2}}\ dS. \tag{18.7.6}$$

We evaluate the integral in (18.7.6). To do this, introduce spherical coordinates in the standard position. We have $dS = a^2 \sin(\phi) d\phi\ d\theta$ and $z = a\cos(\phi)$. So (18.7.6) becomes

$$\int\limits_0^{\pi} \int\limits_0^{2\pi} \frac{(b - a\cos\phi)a^2 \sin\phi}{(a^2 + b^2 - 2ab\cos\phi)^{(3/2)}}\ d\theta\ d\phi;$$

which reduces, after the first integration with respect to θ, to

$$2\pi a^2 \int\limits_0^{\pi} \frac{(b - a\cos\phi)\sin\phi\ d\phi}{(a^2 + b^2 - 2ab\cos\phi)^{3/2}} \tag{18.7.7}$$

Let $u = \cos(\phi)$, hence $du = -\sin(\phi)\ d\phi$. This transforms (18.7.7) into

$$-2\pi a^2 \int\limits_1^{-1} \frac{(b - au)\ du}{(a^2 + b^2 - 2abu)^{3/2}}. \tag{18.7.8}$$

Then we make a second substitution, $v = a^2 + b^2 - 2abu$.
As you may check, this changes (18.7.8) into

$$\frac{2\pi a^2}{4ab^2} \int\limits_{(b-a)^2}^{(b+a)^2} \frac{v + b^2 - a^2}{v^{3/2}}\ dv \tag{18.7.9}$$

Write the integrand as the sum of $1/\sqrt{v}$ and $(b^2 - a^2)/v^{3/2}$, and use the Fundamental Theorem of Calculus, to show that (18.7.8) equals $4\pi a^2/b^2$.

Combining this with (18.7.9) shows that

$$\mathbf{E}(B) = \frac{\sigma}{4\pi\epsilon_0}\frac{4\pi a^2}{b^2}\mathbf{k} = \frac{Q}{4\pi\epsilon_0 b^2}\mathbf{k}.$$

◇

The result in this example, $Q/(4\pi\epsilon_0 b^2)\mathbf{k}$ is the same as if all the charge Q were at the center of the sphere. In other words, a uniform charge on a sphere acts on external particles as though the whole charge were placed at its center. This was discovered for the gravitational field by Newton and proved geometrically in his *Principia* of 1687.

Using Flux and Symmetry to Find E

We included Example 1 for two reasons. First, it reviews some integration techniques. Second, it will help you appreciate a much simpler way to find the field $\mathbf{E}$ due to a charge distribution.

Picture a charge Q distributed outside the region bound by a surface S, as in Figure 18.7.5.

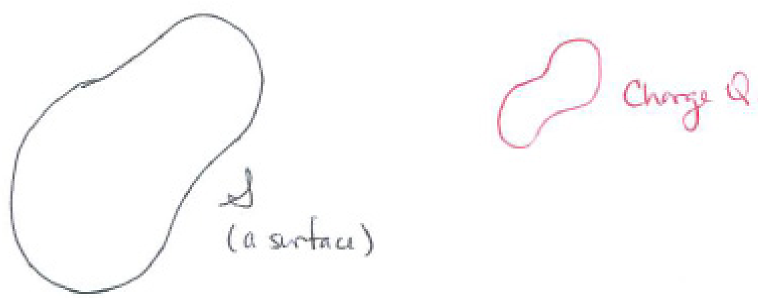


Figure 18.7.5:

The flux of $\mathbf{E}$ associated with a point charge q over a closed surface $\mathcal{S}$ is

$$\int_{\mathcal{S}} \mathbf{E}(P)\cdot\mathbf{n}\, dS = \int_{\mathcal{S}} \frac{\widehat{\mathbf{r}}\cdot\mathbf{n}}{4\pi\epsilon_0 r^2}\, dS = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{S}} \frac{\mathbf{r}\cdot\mathbf{n}}{r^2}\, dS.$$

As we saw in Section 18.5 the integral is 4π when the charge is inside the solid bounded by the surface and 0 if the charge is outside. (See Exercise 28 in that section). Thus the total flux is q/ϵ_0 if the charge is inside and 0 if it is outside.

Consider a charge Q contained wholly within the region bounded by S. We will find the flux of a total charge Q distributed in a solid R inside a surface $\mathcal{S}$. (See Exercise 6 for the case when the charge is outside $\mathcal{S}$.)

Chop the solid R that the charge occupies into n small regions R_1, R_2, $\dots$, R_n. In region R_i select a point P_i. Let the density of charge at P_i be $\delta(P_i)$. Thus the charge in R_i produces a flux of approximately $\delta(P_i)V_i/\epsilon_0$. Consequently

$$\sum_{i=1}^{n} \frac{\delta(P_i)V_i}{\epsilon_0}$$

estimates the flux produced by Q. Taking limits, we see that

$$\text{Flux across } S \text{ produced by } Q = \int_R \frac{\delta(P_i)}{\epsilon_0}\, dV$$

But $\int_R \delta(P_i)\, dV$ is the total charge Q. Thus we have

$$\text{Flux} = \frac{Q}{\epsilon_0}.$$

Thus we have one of the four fundamental equations of electrostatics:

Gauss' Law

The flux produced by a distribution of charge across a closed surface is the charge Q in the region bounded by the surface divided by ϵ_0.

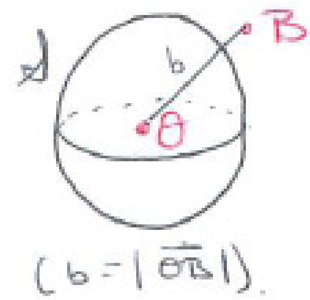


Figure 18.7.6:

The charge outside of S produces no flux across S. (More precisely, the negative flux across S cancels the positive flux.)

Let's illustrate the power of Gauss' Law by applying it to the case in Example 1.

EXAMPLE 2 A charge Q is distributed uniformly on a sphere of radius a. Find the electrostatic field $\mathbf{E}$ at a point B at a distance b from the center of a sphere of radius a, with $b > a$.

SOLUTION We don't need to introduce a coordinate system in Figure 18.7.6. By symmetry, the field at any point P outside the sphere is parallel to the vector $\overrightarrow{CP}$. Moreover, the magnitude of the field is the same for all points at a given distance from the origin C. Call this magnitude, $f(r)$, where r is the distance from C. We want to find $f(b)$.

To do this, imagine another sphere S^*, with center C and radius b, as in Figure 18.7.7.

The flux of $\mathbf{E}$ across S^* is $\int_{S^*} \mathbf{E}\cdot\mathbf{n}\, dS$.

Figure 18.7.7:

But $\mathbf{E} \cdot \mathbf{n}$ is just $f(b)$ since $\mathbf{E}$ and $\mathbf{n}$ are parallel and $\mathbf{E}(P)$ has magnitude $f(b)$ for all points P on S^*. Thus $\int_{S^*} \mathbf{E} \cdot \mathbf{n}\, dS = \int_{S^*} f(b)\, dS = f(b) \int_{S^*} dS = f(b) 4\pi b^2$.

By Gauss' Law

$$\frac{Q}{\epsilon_0} = f(b)(4\pi b^2).$$

That tells us that

$$f(b) = \frac{Q}{4\pi\epsilon_0 b^2}.$$

This is the same result as in Example 1, but compare the work in each case. Symmetry and Gauss' Law provide an easy way to find the electrostatic field due to distribution of charge. $\diamond$

The same approach shows that the field $\mathbf{E}$ produced by the spherical charge in Examples 1 and 2 inside the sphere is $\mathbf{0}$. Let $f(r)$ be the magnitude of $\mathbf{E}$ at a distance r from the center of the sphere. For $r > a$, $f(r) = Q/(4\pi\epsilon_0 r^2)$; for $0 < r < a$, $f(r) = 0$. The graph of f is shown in Figure 18.7.8.

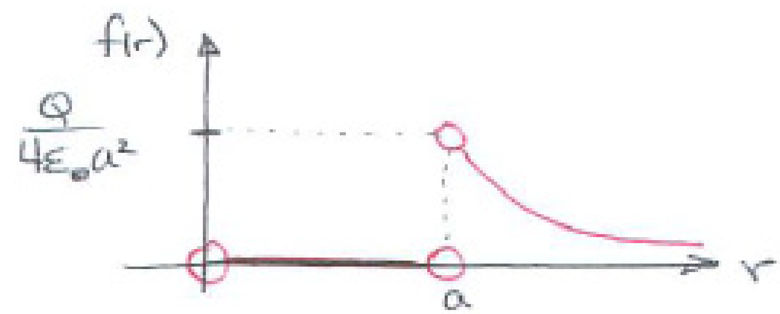


Figure 18.7.8:

If you are curious about $f(a)$ and $f(0)$, see Exercises 8 and 9.

Summary

The field due to a point charge q at a point C is given by the formula $\mathbf{E}(P) = \frac{1}{4\pi\epsilon_0} \frac{q\widehat{\mathbf{r}}}{r^2}$, where $\mathbf{r} = \overrightarrow{OP}$. This field produces a force $q_0\mathbf{E}(P)$ on a charge q_0 located at P.

The field due to a distribution of charge is obtained by an integration over a surface of solid region, depending where this charge is distributed.

We showed that a charge Q outside a surface produces a net flux of zero across the surface. However the flux produced by a charge within the surface is simply Q/ϵ_0. That is Gauss's Law.

We used Gauss's Law to find the field produced by a spherical distribution of charge.

EXERCISES for Section 18.7 *Key:* R–routine, M–moderate, C–challenging

1.[R] The charge q is positive and produces the electrostatic field $\mathbf{E}$. In what direction does $\mathbf{E}$ point at a charge q_0 that is (a) positive and (b) negative?

2.[R] Fill in the omitted details in the calculation in Exercise 1.

3.[R] Describe to a friend who knows no physics the field $\mathbf{E}$ produced by a point charge q.

4.[R] State Gauss's Law aloud several times.

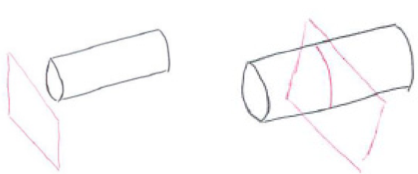
5.[R] Why do you think that the constant k was replaced by $1/4\pi\epsilon_0$. NOTE: Later we will see why it is convenient to have ϵ_0 in the denominator.

6.[R] Show that a charge Q distributed in a solid region R outside a closed surface $\mathcal{S}$ induces zero-flux across $\mathcal{S}$.

7.[R] A charge is distributed uniformly over an infinite plane. For any part of this surface of area A the charge is kA, where k is a constant. Find the field $\mathbf{E}$ due to the charge at any point P not in the plane.

(a) Use symmetry to say as much as you can about it. Be sure to discuss its direction.

(b) Show that the magnitude is constant by applying Gauss's Theorem to a cylinder whose axis is perpendicular to the plane and which does not intersect the plane.



(a) (b)

Figure 18.7.9:

(c) Find the magnitude of $\mathbf{E}$ by applying Gauss's Theorem to the cylinder in Figure 18.7.9(b). Let the area of the circular cross section be A and the area of its curved side be B.

8.[R] Find the field $\mathbf{E}$ of the charge in Example 1 at a point on the surface of the sphere. Why is Gauss's Law not applicable here? HINT: Let the point be $(0,0,a)$.

9.[R] Find the field $\mathbf{E}$ of the charge in Example 1 at the center of the sphere. HINT: Use symmetry, don't integrate.

10.[R] Complete the graph in Figure 18.7.8. That is, fill in the function values corresponding to $r=0$ and $r=a$.

11.[R] A charge is distributed uniformly along an infinite straight wire. The charge on a section of length l is kl. Find the field $\mathbf{E}$ due to this charge.

(a) Use symmetry to say as much as you can about the direction and magnitude of $\mathbf{E}$.

(b) Find the magnitude by applying Gauss's Law to the cylinder of radius r and height h shown in Figure 18.7.10

(c) Find the force directly by an integral over the line, as in Example 1.

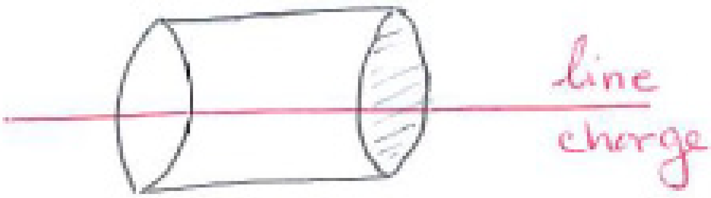


Figure 18.7.10:

12.[R] Figure 18.7.11(a) shows four surfaces. Inside S_1 is a total charge Q_1, and inside S_2 is a total charge Q_2. Find the total flux across each of the four surfaces.

(a) (b)

Figure 18.7.11:

13.[R] Imagine that there is a uniform distribution of charge Q throughout a ball of radius a. Use Gauss' Law to find the electrostatic field $\mathbf{E}$ produced by this charge

(a) at points outside the ball,

(b) at points inside the ball.

14.[R] Let $f(r)$ be the magnitude of the field in Exercise 13 at a distance r from the center of the ball. Graph $f(r)$ for $r \geq 0$.

15.[R] A charge Q lies partly inside a closed surface S and partly outside. Let Q_1 be the amount inside and Q_2 the amount outside, as in Figure 18.7.11(b). What is the flux across $\mathcal{S}$ of the charge Q?

16.[R] In Exercise 11 you found the field $\mathbf{E}$ due to a charge uniformly spread on an infinite line. If the charge density is λ, $\mathbf{E}$ at a point at a distance a from the line is $(\lambda/(2\pi a\epsilon_0))\,\mathbf{j}$.
Now assume that the line occupies only the right half of the x-axis, $[0,\infty)$.

(a) Using the result in Exercise 11, show that the $\mathbf{j}$-component of $\mathbf{E}(0,a)$ is $(\lambda/4\pi a\epsilon_0)\mathbf{j}$.

(b) By integrating over $[0,\infty)$, show that the $\mathbf{i}$-component of $\mathbf{E}$ at $(0,a)$ is $\lambda/(4\pi a\epsilon_0)\mathbf{i}$.

(c) What angle does $\mathbf{E}(0,a)$ make with the y-axis?

(d) Why is Gauss' Law of no use in determining the $\mathbf{i}$-component of $\mathbf{E}$ in this case.

17.[M] We showed that $\mathbf{E}(P) = \frac{\delta}{4\pi\epsilon_0}\int_R \frac{\widehat{\mathbf{r}}}{r^2}\,dV$ if the charge density is constant. Find the corresponding integral for $\mathbf{E}(P)$ when the charge density varies.

18.[C] In Example 1, we used an integral to find the electrostatic field outside a uniformly charged sphere. Carry out similar calculation to find the field inside the sphere. Hint: Is the square root of $(b-a)^2$ still $b-a$?

19.[C] Use the approach in Example 2 to find the electrostatic field *inside* a uniformly charged sphere.

20.[C] Graph the magnitude of the field in Example 1 as a function of the distance from the center of the sphere. This will need the results of Exercises 18 and 19.

21.[C] Find the field $\mathbf{E}$ in the Exercise 7 by integrating over the whole (infinite) plane. (Do not use Gauss's Theorem.)

18.8 Expressing Vector Functions in Other Coordinate Systems

We have expressed the gradient, divergence, and curl in terms of rectangular coordinates. However, students who apply vector analysis in engineering and physics courses will see functions expressed in polar, cylindrical, and spherical coordinates. This section shows how those expressions are found.

The Gradient in Polar Coordinates

Let $g(r,\theta)$ be a scalar function expressed in polar coordinates. Its gradient has the form $A(r,\theta)\widehat{\mathbf{r}} + B(r,\theta)\widehat{\theta}$, where $\widehat{\mathbf{r}}$ and $\widehat{\theta}$ are the unit vectors shown in Figure 18.8.1. The unit "radial vector" $\widehat{\mathbf{r}}$ points in the direction of increasing r. The unit "tangential vector" $\widehat{\theta}$ points in the direction determined by increasing θ. Note that $\widehat{\theta}$ is tangent to the circle through (r,θ) with center at the pole.

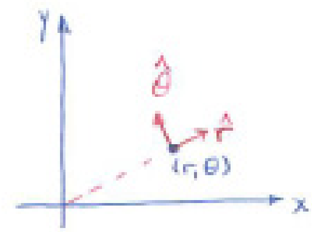


Figure 18.8.1:

Our goal is to find $A(r,\theta)$ and $B(r,\theta)$, which wedenote simply as A and B.

One might guess, in analogy with rectangular coordinates, that $A(r,\theta)$ would be $\partial g/\partial r$ and $B(r,\theta)$ would be $\partial g/\partial\theta$. That guess is part right and part wrong, for we will show that

$$\boxed{\operatorname{grad} g = \frac{\partial g}{\partial r}\widehat{\mathbf{r}} + \frac{1}{r}\frac{\partial g}{\partial\theta}\widehat{\theta}} \tag{18.8.1}$$

We reserve the use of ∇ for rectangular coordinates, and use grad in all other coordinate systems.

Note the appearance of $1/r$ in the $\widehat{\theta}$ component.

One way to obtain (18.8.1) is labor-intensive and not illuminating: express g, $\widehat{\mathbf{r}}$, and $\widehat{\theta}$ in terms of x, y, $\mathbf{i}$, $\mathbf{j}$ and use the formula for gradient in terms of rectangular coordinates, then translate back to polar coordinates. This approach, whose only virtue is that it offers good practice applying the chain rule for partial derivatives, is outlined in Exercises 17 and 18.

We will use a simpler way, that easily generalizes to the cylindrical and spherical coordinates. It exploits the connection between a gradient and directional derivative of g at a point P in the direction $\mathbf{u}$. In particular, it shows why the coefficient $1/r$ appears in (18.8.1).

Recall that if $\mathbf{u}$ is a unit vector, the directional derivative of g in the direction $\mathbf{u}$ is the dot product of $\operatorname{grad} g$ with $\mathbf{u}$:

$$D_{\mathbf{u}}g = \operatorname{grad} g \cdot \mathbf{u}.$$

In particular,

$$D_{\widehat{\mathbf{r}}}g = (A\widehat{\mathbf{r}} + B\widehat{\theta}) \cdot \widehat{\mathbf{r}} = A$$

and

$$D_{\widehat{\theta}}g = (A\widehat{\mathbf{r}} + B\widehat{\theta}) \cdot \widehat{\theta} = B.$$

So all we need to do is find $D_{\widehat{\mathbf{r}}}g$ and $D_{\widehat{\theta}}g$.

First,

$$D_{\widehat{\mathbf{r}}}(g) = \lim_{\Delta r \to 0} \frac{g(r+\Delta r, \theta) - g(r,\theta)}{\Delta r} = \frac{\partial g}{\partial r}.$$

So $A(r,\theta) = \partial g/\partial r(r,\theta)$. That explains the expected part of (18.8.1).

Now we will see why B is *not* simply the partial derivation of g with respect to θ.

If we want to estimate a directional derivative at P of g in the direction $\mathbf{u}$ we pick a nearby point Q a distance Δs away in the direction of $\mathbf{u}$ and form the quotient

$$\frac{g(Q) - g(P)}{\Delta s} \tag{18.8.2}$$

Then we take the limit of (18.8.2) as $\Delta s \to 0$.

Now let $\mathbf{u}$ be $\widehat{\theta}$, and let's examine (18.8.2) in the case hwere $P = (r,\theta)$ and $Q = (r, \theta + \Delta\theta)$. The numerator in (18.8.2) is

$$g(r, \theta + \Delta\theta) - g(r,\theta).$$

We draw a picture to find Δs, as in Figure 18.8.2.

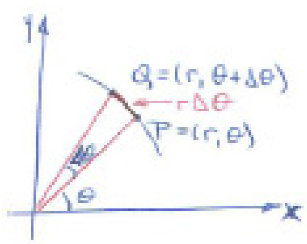


Figure 18.8.2:

The distance between P and Q is *not* $\Delta\theta$. Rather it is approximately $r\Delta\theta$ (when $\Delta\theta$ is small). That tells us that Δs in (18.8.2) is not $\Delta\theta$ but $r\Delta\theta$. Therefore

$$D_{\theta}g = \lim_{\Delta\theta \to 0} \frac{g(r, \theta+\Delta\theta) - g(r,\theta)}{r\Delta\theta} = \frac{1}{r} \lim_{\Delta\theta \to 0} \frac{g(r,\theta+\Delta\theta) - g(r,\theta)}{\Delta\theta} = \frac{1}{r}\frac{\partial g}{\partial \theta}.$$

Note $r\ \Delta\theta$ in the denominator.

That is why there is a $1/r$ in the formula (18.8.1) for the gradient of g. It occurs because a change $\Delta\theta$ in the parameter θ causes a point to move approximately the distance $r\Delta\theta$.

Divergence in Polar Coordinates

The divergence of $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ is simply $\partial P/\partial x + \partial Q/\partial y$. But what is the divergence of a vector field described in polar coordinates, $\mathbf{G}(r,\theta) = A(r,\theta)\widehat{\mathbf{r}} + B(r,\theta)\widehat{\theta}$. By now you are on guard, $\nabla \cdot \mathbf{G}$ is *not* the sum of $\partial A/\partial r$ and $\partial B/\partial \theta$).

To find $\nabla \cdot \mathbf{G}$, use the relation between $\nabla \cdot \mathbf{G}$ at $P = (r,\theta)$ and the flux across a small curve C that surrounds P.

$$\nabla \cdot \mathbf{G} = \lim_{\text{length of } C \to 0} \frac{\oint_C \mathbf{G} \cdot \mathbf{n}\, ds}{\text{Area within } C} \tag{18.8.3}$$

Note that (18.8.3) provides a coordinate-free description of divergence in the plane.

We are free to choose the small closed curve C to make it easy to estimate the flux across it. A curve C that corresponds to small changes Δr and $\Delta\theta$ is convenient is shown in Figure 18.8.3. We will use (18.8.3) to find the divergence at $P = (r, \theta)$. Now, P is not inside C; rather it is on C. However, since $\mathbf{G}$ is continuous, $\mathbf{G}(P)$ is the limit of values of $\mathbf{G}$ at points inside, so we may use (18.8.3).

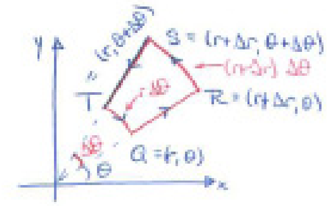


Figure 18.8.3: C is the curve $QRSTQ$

To estimate the flux across C, we estimate the flux across each of the four parts of the curve. Because these sections are short when Δr and $\Delta\theta$ are small, we may estimate the integral over each part by multiplying the value of the integrand at any point of the section (even at an end point) by the length of the section. As usual, $\widehat{\mathbf{n}}$ denotes an exterior unit vector perpendicular to C.

On QR and ST, $B\theta$ contributes to the flux (on RS and TQ it does not since $\mathbf{n} \cdot \theta$ is 0). On QR, θ is parallel to $\mathbf{n}$, as shown in Figure 18.8.4.

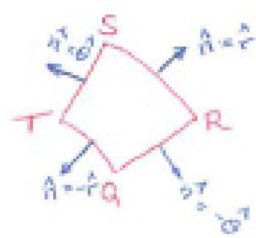


Figure 18.8.4:

However, on ST it points in the opposite direction, $\widehat{\theta} \cdot \widehat{\mathbf{n}}$ is -1. So, across ST, the flux contributed by $B\widehat{\theta}$ is approximately

$$(B\widehat{\theta} \cdot \widehat{\mathbf{n}})\Delta r = -B(r, \theta)\Delta r.$$

(We would get a better estimate by using $B(r + \frac{\Delta r}{2}, \theta)$ but $B(r, \theta)$ is good enough since B is continuous.)

On QR, $\widehat{\theta}$ and $\widehat{\mathbf{n}}$ point in almost the same direction, hence $\theta \cdot \widehat{\mathbf{n}}$ is close to 1 when $\Delta\theta$ is small. So on ST, $B\widehat{\theta}$ contributes approximately $B(r, \theta + \Delta\theta)\Delta r$ to the flux.

All told, the total contribution of $B\theta$ to the flux across C is

$$B(r, \theta + \Delta\theta)\Delta r - B(r, \theta)\Delta r \tag{18.8.4}$$

The contribution of $A\widehat{\mathbf{r}}$ to the flux is negligible on QR and ST because there $\widehat{\mathbf{r}}$ and $\widehat{\mathbf{n}}$ are perpendicular. On TQ, $\widehat{\mathbf{r}}$ and $\widehat{\mathbf{n}}$ point in almost directly opposite directions, hence $\widehat{\mathbf{r}} \cdot \widehat{\mathbf{n}}$ is near -1. The flux of $A\widehat{\mathbf{r}}$ there, is approximately

$$A(r, \theta)(\widehat{\mathbf{r}} \cdot \widehat{\mathbf{n}})r\Delta\theta = -A(r, \theta)r\Delta\theta. \tag{18.8.5}$$

On RS, which has radius $r + \Delta r$, $\widehat{\mathbf{r}}$ and $\widehat{\mathbf{n}}$ are almost identical, hence $\widehat{\mathbf{r}} \cdot \widehat{\mathbf{n}}$ is near 1. The contribution on RS, which has radius $r + \Delta r$ is approximately

$$A(r + \Delta r, \theta)(r + \Delta r)\Delta\theta. \tag{18.8.6}$$

Combining (18.8.4), (18.8.5) and (18.8.6), we see that the limit in (18.8.3) is the sum of two limits:

$$\lim_{\Delta r, \Delta\theta \to 0} \frac{A(r + \Delta r, \theta)(r + \Delta r)\Delta\theta - A(r, \theta)r\Delta\theta}{r\Delta r\Delta\theta} \tag{18.8.7}$$

The area within C is approximately, $r\Delta r\Delta\theta$.

and

$$\lim_{\Delta r,\Delta\theta\to 0}\frac{B(r,\theta+\Delta\theta)\Delta r - B(r,\theta)\Delta r}{r\Delta r\Delta\theta} \tag{18.8.8}$$

The first limit (18.8.7) equals

$$\lim_{\Delta r,\Delta\theta\to 0}\frac{1}{r}\frac{(r+\Delta r)A(r+\Delta r,\Delta\theta) - rA(r,\theta)}{\Delta r},$$

which is

$$\frac{1}{r}\frac{\partial(rA)}{\partial r}.$$

Note that r appears in the coefficient, $1/r$, and also in the function, rA, being differentiated.

The second limit (18.8.8) equals

$$\lim_{\Delta r,\Delta\theta\to 0}\frac{1}{r}\frac{B(r,\theta+\Delta\theta) - B(r,\theta)}{\Delta\theta},$$

hence is

$$\frac{1}{r}\frac{\partial B}{\partial\theta}.$$

Here r appears only once, in the coefficient.

Note the use of div, not $\nabla\cdot$.

All told, we have the desired divergence formula:

$$\boxed{\operatorname{div}(A\widehat{\mathbf{r}} + B\theta) = \frac{1}{r}\frac{\partial(rA)}{\partial r} + \frac{1}{r}\frac{\partial B}{\partial\theta}. \qquad (18.8.9)}$$

Curl in the Plane

The curl of $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j} + 0\mathbf{k}$, a vector field in the plane, is given by the formula

$$\mathbf{curl\,F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}.$$

What is the formula for the curl when the field is described in polar coordinates: $\mathbf{G}(r,\theta) = A(r,\theta)\widehat{\mathbf{r}} + B(r,\theta)\widehat{\mathbf{n}}$? To find out we will reason as we did with divergence. This time we use

$$(\mathbf{curl\,G})\cdot\widehat{\mathbf{n}} = \lim_{\text{length of } C\to 0}\frac{\oint_C \mathbf{G}\cdot\mathbf{k}\,ds}{\text{Area bounded by } C}.$$

where C is a closed curve around a fixed point in the (r,θ) plane, and the limit is taken as the length of C approaches 0. The curl is evaluated at a fixed point, which is on or within C.

See (18.6.9) on page 1348.

We compute the circulation of $\mathbf{G} = A\widehat{\mathbf{r}} + B\theta$ around the same curve used in the derivation of divergence in polar coordinates.

On TQ and RS, $A\widehat{\mathbf{r}}$, being perpendicular to the curve, contributes nothing to the circulation of $\mathbf{G}$ around C. On QR it contributes approximately

$$A(r,\theta)(\widehat{\mathbf{r}}\cdot\mathbf{T})\Delta r = A(r,\theta)\Delta r.$$

On ST, since there $\widehat{\mathbf{r}}\cdot\mathbf{T} = -1$, it contributes approximately

$$A(r,\theta+\Delta\theta)(\mathbf{r}\cdot\mathbf{T})\Delta r = -A(r,\theta+\Delta\theta)\Delta r.$$

A similar computation shows that $B\widehat{\theta}$ contributes to the total circulation approximately

$$B(r+\Delta r,\theta)(r+\Delta r)\Delta\theta - B(r,\theta)r\Delta\theta.$$

Therefore $(\nabla\times\mathbf{G}\cdot)\mathbf{k}$ in the sum of two limits:

$$\lim_{\Delta r,\Delta\theta\to 0}\frac{A(r,\theta)\Delta r - A(r,\theta+\Delta\theta)\Delta r}{r\Delta r\Delta\theta} = -\frac{1}{r}\frac{\partial A}{\partial\theta}$$

and

$$\lim_{\Delta r,\Delta\theta\to 0}\frac{B(r+\Delta r,\theta)(r+\Delta r)\Delta\theta - B(r,\theta)r\Delta\theta}{r\Delta r\Delta\theta} = \frac{1}{r}\frac{\partial(rB)}{\partial r}.$$

All told, we have

Note the use of **curl**, not $\nabla\times$.

$$\mathbf{curl}(A\widehat{\mathbf{r}} + B\theta) = \left(-\frac{1}{r}\frac{\partial A}{\partial\theta} + \frac{1}{r}\frac{\partial(rB)}{\partial r}\right)\mathbf{k}. \qquad (18.8.10)$$

EXAMPLE 1 Find the divergence and curl of $\mathbf{F} = r\theta^2\widehat{\mathbf{r}} + r^3\tan(\theta)\theta$.
SOLUTION The calculations are direct applications of (18.8.9) and (18.8.10). First, the divergence:

$$\begin{aligned}\operatorname{div}\mathbf{F} &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\cdot r\theta^2\right) + \frac{1}{r}\frac{\partial}{\partial\theta}\left(r^3\tan(\theta)\right)\\ &= \frac{1}{r}\left(2r\theta^2\right) + \frac{1}{r}\left(r^3\sec^2(\theta)\right) = 2\theta^2 + r^2\sec^2(\theta).\end{aligned}$$

And, the curl:

$$\begin{aligned}\mathbf{curl}\,\mathbf{F} &= \left(-\frac{1}{r}\frac{\partial}{\partial\theta}\left(r\theta^2\right)+\frac{1}{r}\frac{\partial}{\partial r}\left(r\cdot r^3\tan(\theta)\right)\right)\mathbf{k}\\ &= \left(-\frac{1}{r}(2r\theta)+\frac{1}{r}\left(4r^3\tan(\theta)\right)\right)\mathbf{k}=\left(-2\theta+4r^2\tan(\theta)\right)\mathbf{k}.\end{aligned}$$

◇

Cylindrical Coordinates

In cylindrical coordinates the gradient of $g(r,\theta,z)$ is

$$\operatorname{grad} g = \frac{\partial g}{\partial r}\widehat{\mathbf{r}}+\frac{1}{r}\frac{\partial g}{\partial\theta}\widehat{\theta}+\frac{\partial g}{\partial z}\widehat{\mathbf{z}} \qquad (18.8.11)$$

Here $\widehat{\mathbf{z}}$ is the unit vector in the positive z direction, denoted $\mathbf{k}$ in Chapter 14. Note that (18.8.11) differs from (18.8.1) only by the extra term $(\partial g/\partial z)\widehat{\mathbf{z}}$. You can obtain (18.8.11) by computing directional derivatives of g along $\widehat{\mathbf{r}}$, $\widehat{\theta}$, and $\widehat{\mathbf{z}}$. The derivation is similar to the one that gave us the formula for the gradient of $g(r,\theta)$.

The divergence of $\mathbf{G}(r,\theta,z)=A\widehat{\mathbf{r}}+B\widehat{\theta}+C\widehat{\mathbf{z}}$ is given by the formula

$$\operatorname{div}\mathbf{G} = \frac{1}{r}\frac{\partial(rA)}{\partial r}+\frac{\partial B}{\partial\theta}+\frac{\partial(rC)}{\partial z}. \qquad (18.8.12)$$

Note that the partial derivatives with respect to r and z are similar in that the factor r is present in both $\partial(rA)/\partial r$ and $\partial(rC)/\partial r$. You can obtain (18.8.12) by using the relation between $\nabla\cdot\mathbf{G}$ and the flux across the small surface determined by small changes Δr, $\Delta\theta$, and Δz.

The curl of $\mathbf{G}=A\widehat{\mathbf{r}}+B\theta+C\widehat{\mathbf{z}}$ is given by a formal determinant:

$$\mathbf{curl}\,\mathbf{G} = \frac{1}{r}\begin{vmatrix}\widehat{\mathbf{r}} & r\widehat{\theta} & \mathbf{k}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z}\\ A & rB & C\end{vmatrix} \qquad (18.8.13)$$

To obtain this formula consider the circulation around three small closed curves lying in planes perpendicular to $\widehat{\mathbf{r}}$, $\widehat{\theta}$ and $\mathbf{k}$.

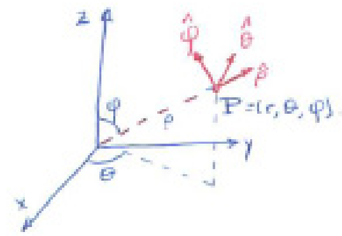


Figure 18.8.5:

Spherical Coordinates

In mathematics texts, spherical coordinates are denoted ρ, ϕ, θ. In physics and engineering a different notation is standard. There ρ is replaced by r, θ is the angle with z-axis, and ϕ plays the role of the mathematicians' θ, switching the roles of ϕ and θ. The formulas we state are in the mathematicians' notation.

The three basic unit vectors for spherical coordinates are denoted ρ, ϕ, θ. For instance, ρ points in the direction of increasing ρ. See Figure 18.8.5. Note that, at the point P, ϕ and θ are tangent to the sphere through P and center at the origin, while ρ is perpendicular to that sphere. Also, any two of ρ, ϕ, θ are perpendicular.

To obtain the formulas for $\nabla \cdot \mathbf{G}$ and $\nabla \times \mathbf{G}$, we would use the region corresponding to small changes $\Delta\rho$, $\Delta\phi$, and $\Delta\theta$, shown in Figure 18.8.6. That computation yields the following formulas:

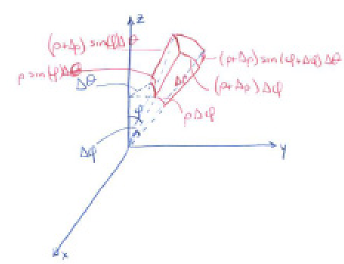


Figure 18.8.6:

If $g(\rho, \phi, \theta)$ is a scalar function,

$$\operatorname{grad} g = \frac{\partial g}{\partial \rho}\rho + \frac{1}{\rho}\frac{\partial g}{\partial \phi}\phi + \frac{1}{\rho\sin(\phi)}\frac{\partial g}{\partial \theta}\theta. \tag{18.8.14}$$

If $\mathbf{G}(\rho, \phi, \theta) = A\rho + B\phi + C\theta$

$$\operatorname{div} \mathbf{G} = \frac{1}{\rho^2}\frac{\partial(\rho^2 A)}{\partial r}) + \frac{1}{\rho\sin(\phi)}\frac{\partial(\sin(\phi)B)}{\partial \phi} + \frac{1}{\rho\sin(\phi)}\frac{\partial C}{\partial \theta}. \tag{18.8.15}$$

and

$$\begin{aligned}\mathbf{curl}\,\mathbf{G} &= \frac{1}{\rho}\left(\frac{1}{\sin(\phi)}\frac{\partial(\sin(\phi))C)}{\partial \phi} - \frac{1}{\rho\sin(\phi)}\frac{\partial B}{\partial \theta}\right)\rho \\ &\quad + \frac{1}{\rho}\left(\frac{1}{\sin(\phi)}\frac{\partial A}{\partial \theta} - \frac{\partial(\rho C)}{\partial \rho}\right)\phi + \frac{1}{\rho}\left(\frac{\partial(\rho B)}{\partial \rho} - \frac{\partial A}{\partial \phi}\right)\theta\end{aligned}$$

Each of these can be obtained by the method we used for polar coordinates. In each case, keep in mind that the change in ϕ or θ is not the same as the distance the corresponding point moves. However, a change in ρ is the same as the distance the corresponding point moves. For instance, the distance between (ρ, ϕ, θ) and $(\rho, \phi + \Delta\phi, \Delta\theta)$ is approximately $\rho\Delta\phi$ and the distance between (ρ, ϕ, θ) and $(\rho, \phi, \theta + \Delta\theta)$ is approximately $\rho\sin(\phi)\Delta\theta$.

An Application to Rotating Fluids

Consider a fluid rotating in a cylinder, for instance, in a centrifuge. If it rotates as a rigid body, then its velocity at a distance r from the axis of rotation has the form

$$\mathbf{G}(r, \theta) = cr\theta,$$

where c is a positive constant.

Then

$$\mathbf{curl}\,\mathbf{G} = \frac{1}{r}\frac{\partial(cr^2)}{\partial r}\mathbf{k} = 2c\mathbf{k}.$$

The curl is independent of r. That means that an imaginary paddle held with its axis held in a fixed position would rotate at the same rate no matter where it is placed.

Now consider the more general case with

$$\mathbf{G}(r, \theta) = cr^n\theta,$$

and n is an integer. Now

$$\mathbf{curl}\,\mathbf{G} = \frac{1}{r}\frac{\partial(cr^{n+1})}{\partial r}\mathbf{k} = c(n+1)r^{n-1}\mathbf{k}.$$

We just considered the case $n = 1$. If $n > 1$, the curl increases as r increases. The paddle wheel rotates faster if placed farther from the axis of rotation. The direction of rotation is the same as that of the fluid, counterclockwise.

Next consider the case $n = -2$. The speed of the fluid *decreases* as r increases. Now

$$\mathbf{curl}\,\mathbf{G} = c(-2+1)r^{-2-1}\mathbf{k} = -cr^{-3}\mathbf{k}.$$

The minus sign before the coefficient c tells us that the paddle wheel spins *clockwise* even though the fluid rotates counterclockwise. The farther the paddle wheel is from the axis, the slower it rotates.

Summary

We expressed gradient, divergence, and curl in several coordinate systems. Even though the basic unit vectors in each system may change direction from point to point, they remain perpendicular to each other. That simplified the computation of flux and circulation. The formulas are more complicated than those in rectangular coordinates because tha amount a parameter changes is not the same as the distance the corresponding point moves.

EXERCISES for Section 18.8 *Key:* R–routine, M–moderate, C–challenging

In Exercises 1 through 4 find and draw the gradient of the given functions of (r, θ) at $(2, \pi/4)$.

1.[R] r

2.[R] $r^2\theta$

3.[R] $e^{-r}\theta$

4.[R] $r^3\theta^2$

In Exercises 5 through 8 find the divergence of the given function

5.[R] $5\widehat{\mathbf{r}} + r^2\theta\widehat{\theta}$

6.[R] $r^3\theta\widehat{\mathbf{r}} + 3r\theta\widehat{\theta}$

7.[R] $r\widehat{\mathbf{r}} + r^3\widehat{\theta}$

8.[R] $r\sin(\theta)\widehat{\mathbf{r}} + r^2\cos(\theta)\widehat{\theta}$

In Exercises 9 through 12 compute the curl of the given function.

9.[R] $r\widehat{\theta}$

10.[R] $r^3\theta\widehat{\mathbf{r}} + e^r\widehat{\theta}$

11.[R] $r\cos(\theta)\widehat{\mathbf{r}} + r\theta\widehat{\theta}$

12.[R] $1/r^3\widehat{\theta}$

13.[R] Find the directional derivative of $r^2\theta^3$ in the direction

(a) $\widehat{\mathbf{r}}$

(b) $\widehat{\theta}$

(c) $\mathbf{i}$

(d) $\mathbf{j}$

14.[R] What property of rectangular coordinates makes the formulas for gradient, divergence, and curl in those coordinates relatively simple?

15.[R] Estimate the flux of $r\theta vrhat = r^2\theta^3\widehat{\theta}$ around the circle of radius 0.01 with center at $(r, \theta) = (2, \pi/6)$.

16.[R] Estimate the circulation of the field in the preceding exercise around the same circle.

When translating between rectangular and polar coordinates, it may be necessary to express $\widehat{\mathbf{r}}$ and $\widehat{\theta}$ in terms of $\mathbf{i}$ and $\mathbf{j}$ and also $\mathbf{i}$ and $\mathbf{j}$ in terms of $\mathbf{r}$ and $\widehat{\theta}$. Exercise 17 and 18 concern this matter.

17.[R] Let (r, θ) be a point that has rectangular coordinates (x, y).

(a) Show that $\widehat{\mathbf{r}} = \cos(\theta)\mathbf{i} + \sin(\theta)\mathbf{j}$, which equals $x/\sqrt{x^2+y^2}\mathbf{i} + y/\sqrt{x^2+y^2}\mathbf{j} = \frac{x\mathbf{i}+y\mathbf{j}}{\sqrt{x^2+y^2}}$

(b) Show that $\theta = -\sin(\theta)\mathbf{i} + \cos(\theta)\mathbf{j}$, which equals $-y/\sqrt{x^2+y^2}\mathbf{i} + x/\sqrt{x^2+y^2}\mathbf{j}$.

(c) Draw a picture to accompany the calculations done in (a) and (b).

So we have $\widehat{\mathbf{r}}$ and θ in terms of $\mathbf{i}$ and $\mathbf{j}$:

$$\begin{cases} \widehat{\mathbf{r}} &= \frac{x\mathbf{i}+y\mathbf{j}}{\sqrt{x^2+y^2}} \\ \theta &= \frac{-y\mathbf{i}+x\mathbf{j}}{\sqrt{x^2+y^2}} \end{cases} \tag{18.8.16}$$

18.[R] Show that if (x, y) has polar coordinates (r, θ), then

$$\begin{cases} \mathbf{i} = \cos(\theta)\theta\widehat{\mathbf{r}} - \sin(\theta)\theta \\ \mathbf{j} = \sin(\theta)\theta\widehat{\mathbf{r}} + \cos(\theta)\theta \end{cases}$$

by solving the simultaneous equations (18.8.16) in the preceding exercise for $\mathbf{i}$ and $\mathbf{j}$.

In exercises 19 through 22

I. find the gradient of the given function, using the formula for gradient in rectangular coordinates,

II. find it by first expressing the function in polar coordinates and again for gradient in polar coordinates. (18.8.1),

show that the two results agree.

19.[R] $x^2 + y^2$

20.[R] $\sqrt{x^2+y^2}$

21.[R] $3x + 2y$

22.[R] $x/\sqrt{x^2+y^2}$

In Exercises 23 through 26

I. find the gradient of the given function, using its formula in polar coordinates, that is (18.8.1),

II. find it by first expressing the function in rectangular coordinates,

III. show that the two results agree.

23.[R] r^2

24.[R] $r^2\cos(\theta)$

25.[R] $r\sin(\theta)$

26.[R] e^r

In Exercise 27 and 28

I. find the divergence of the given vector field in rectangular coordinates,

II. find it by first expressing the function in polar coordinates and using (18.8.9),

III. show that the results agree.

27.[R] $x^2\mathbf{i} + y^2\mathbf{j}$

28.[R] $xy\mathbf{i}$

In Exercises 29 and 30

I. find the curl of the given vector field in rectangular coordinates,

II. find it by first expressing the function in polar coordinates and using (18.8.10),

III. show that the two results agree.

29.[R] $xy\mathbf{i} + x^2y^2\mathbf{j}$

30.[R] $(x/\sqrt{x^2+y^2})\mathbf{i}$

The next two exercises are useful in developing the formula for the gradient in cylindrical and spherical coordinates.

31.[R] Approximately how far is it from the points (r,θ,z) to

(a) $(r+\Delta r,\theta,z)$,

(b) $(r,\theta+\Delta\theta,z)$,

(c) $(r,\theta,z+\Delta z)$.

32.[R] Approximate the distance from the point (ρ,ϕ,θ) to

(a) $(\rho+\Delta\rho,\phi,\theta)$,

(b) $(\rho,\phi+\Delta\phi,\theta)$,

(c) $(\rho,\phi,\theta+\Delta\theta)$.

33.[M] Using the formulas for the gradient of $g(r,\phi,\theta)$, find the directional derivative of g in the direction

(a) $\widehat{\rho}$,

(b) $\widehat{\phi}$,

(c) $\widehat{\theta}$.

34.[M] Using the formulas for the gradient of $g(r,\theta,z)$, find the directional derivative of g in the direction

(a) $\widehat{\mathbf{r}}$,

(b) θ,

(c) $\mathbf{k}$.

35.[M] Without using the formula for the gradient, do Exercise 33.

36.[M] Without using the formula for the gradient, do Exercise 34.

37.[M] Using as few mathematical symbols as you can, state the formula for the divergence of a vector field given relative to $\widehat{\mathbf{r}}$ and θ.

38.[M] Using as few mathematical symbols as you can, state the formula for the curl of a vector field given relative to $\widehat{\mathbf{r}}$ and θ.

39.[M] In the formula for the divergence of $A\widehat{\mathbf{r}} + B\widehat{\theta}$, why do the terms rA and $1/r$ appear in $(1/r)(\partial(rA)/\partial r$ and rA? Explain in detail why $1/r$ appears.

40.[M] Obtain the formula for the gradient in cylindrical coordinates.

41.[M] Obtain the formula for curl in cylindrical coordinates.

42.[M] Obtain the formula for divergence in cylindrical coordinates.

43.[M] Obtain the formula for the gradient in spherical coordinates.

44.[M] Where did we use the fact that $\widehat{r}$ and $\widehat{\theta}$ are perpendicular when developing the expression for divergence in polar coordinates?

45.[M] Obtain the formula for the gradient of $g(r,\theta)$ in polar coordinates by starting with the formula for the gradient of $f(x,y)$ in rectangular coordinates. During the calculations you will have some happy moments as complicated expressions cancel and the identity $\cos^2(\theta) + \sin^2(\theta) = 1$ simplifies expressions. (See Exercise 18.8.16.)
Assume $g(r,\theta) = f(x,y)$, where $x = r\cos(\theta)$ and $y = r\sin(\theta)$. To express $\nabla f = \partial f/\partial x\mathbf{i} + \partial f/\partial y\mathbf{j}$ in terms of polar coordinates, it is necessary to express $\partial f/\partial x$, $\partial f/\partial y$, $\mathbf{i}$, and $\mathbf{j}$ in terms of partial derivative of $g(r,\theta)$ and $\widehat{\mathbf{r}}$ and θ.

(a) Show that $\partial r/\partial x = \cos(\theta)$, $\partial r/\partial y = \sin(\theta)$, $\partial\theta/\partial x = -(\sin(\theta))/r$, $\partial\theta/\partial y = (\cos\theta)/r$.

(b) Use the chain rule to express $\partial f/\partial x$ and $\partial f/\partial y$ in terms of partial derivatives of $g(r,\theta)$.

(c) Recalling the expression of $\mathbf{i}$ and $\mathbf{j}$ in terms of $\widehat{r}$ and $\widehat{\theta}$ in Exercise 18 obtain the gradient of $g(r,\theta)$ in polar coordinates.

46.[M] In Exercise 26 of Section 18.3, we found the divergence of $\mathbf{F} = r^n\widehat{\mathbf{r}}$ using rectangular coordinates. Find the divergence using polar coordinates formally. NOTE: The second way is much easier.

47.[M] In Exercise 6 of Section 18.6 we used rectangular coordinates to show that an irrotational planar central field is symmetric. Use the formula for curl in polar coordinates to obtain the same result. NOTE: This way is much easier.

48.[M] In Exercise 21 in Section 18.4 we used rectangular coordinates to show that an incompressible symmetric central field in the plane must have the form $\mathbf{F}(\mathbf{r}) = (k/r)\widehat{\mathbf{r}}$. Obtain this result using the formula for divergence in polar coordinates.

18.9 Maxwell's Equations

At any point in space there is an electric field $\mathbf{E}$ and a magnetic field $\mathbf{B}$. The electric field is due to charges (electrons and protons) whether stationary or moving. The magnetic field is due to moving charges.

To assure yourself that the magnetic field $\mathbf{B}$ is everywhere, hold up a pocket compass. The magnetic field, produced within the Earth, makes the needle point north.

All of the electrical phenomena and their applications can be explained by four equations, called **Maxwell's equations**. These equations allow $\mathbf{B}$ and $\mathbf{E}$ to vary in time. We state them for the simpler case when B and E are constant: $\partial\mathbf{B}/\partial t = \mathbf{0}$ and $\partial\mathbf{E}/\partial t = \mathbf{0}$. We met the first equation in the previous section. Here is the complete list

I. $\int_S \mathbf{E}\cdot\mathbf{n}\; dS = Q/\epsilon_0$, where S is a surface bounding a spatial region and Q is the change in that region. (Gauss's Law for Electricity)

II. $\oint_C \mathbf{E}\cdot d\mathbf{r} = 0$ for any closed curve C. (Faraday's Law of Induction)

III. $\int_S \mathbf{B}\cdot\mathbf{n}\; dS = 0$ for any surface S that bounds a spatial region. (Gauss's Law for Magnetism)

IV. $\oint_C \mathbf{B}\cdot d\mathbf{r} = \mu_0 \int_S \mathbf{J}\cdot\mathbf{n}\; dS$, where C bounds the surface $\mathcal{S}$ and $\mathbf{J}$ is the electric current flowing through $\mathcal{S}$. (Ampere's Law)

The constants ϵ_0 and μ_0 ("myoo zero") depend on the units used. They will be important in the CIE on Maxwell's Equations.

Each of the four statements about integrals can be translated into information about the behavior of $\mathbf{E}$ or $\mathbf{B}$ at each point.

In derivative or "local" form the four principles read:

I'. $\operatorname{div}\mathbf{E} = q/\epsilon_0$, where q is the charge density (Coulomb's Law)

II'. $\mathbf{curl}\,\mathbf{E} = \mathbf{0}$

III'. $\operatorname{div}\mathbf{B} = 0$

IV'. $\mathbf{curl}\,\mathbf{B} = \mu_0\mathbf{J}$

It turns out that $\frac{1}{\mu_0\epsilon_0}$ equals the square of the speed of light. Why that is justified is an astonishing story told in CIE 23.

Going Back and Forth Between "Local" and "Global."

Examples 1 and 2 show that Gauss's Law is equivalent to Coulomb's.

EXAMPLE 1 Obtain Gauss's Law for Electricity (I) from Coulomb's Law (I').

SOLUTION Let $\mathcal{V}$ be the solid region whose boundary is $\mathcal{S}$. Then

$$\begin{aligned}
\int_{\mathcal{S}} \mathbf{E}\cdot\mathbf{n}\,dS &= \int_{\mathcal{V}} \nabla\cdot\mathbf{E}\,dV && \text{Divergence Theorem}\\
&= \int_{\mathcal{V}} \frac{q}{\epsilon_0}\,dV && \text{Coulomb's Law}\\
&= \frac{1}{\epsilon_0}\int_{\mathcal{V}} q\,dV = \frac{Q}{\epsilon_0}.
\end{aligned}$$

Recall that the total charge in $\mathcal{V}$ is $Q = \int_{\mathcal{V}} q\,dV$. ◇

Does Gauss's law imply Coulomb's law? Example 2 shows that the answer is yes.

EXAMPLE 2 Deduce Coulomb's law (I') from Gauss's law for electricity (I).

SOLUTION Let $\mathcal{V}$ be any spatial region and let $\mathcal{S}$ be its surface. Let Q be the total charge in $\mathcal{V}$. Then

$$\begin{aligned}
\frac{Q}{\epsilon_0} &= \int\limits_{\mathcal{S}} \mathbf{E}\cdot\mathbf{n}\,dS && \text{Gauss's law}\\
&= \int\limits_{\mathcal{V}} \nabla\cdot\mathbf{E}\,dV && \text{Divergence Theorem.}
\end{aligned}$$

On the other hand,

$$Q = \int\limits_{\mathcal{V}} q\,dV,$$

where q is the charge density. Thus

$$\int\limits_{\mathcal{V}} \frac{q}{\epsilon_0}\,dV = \int\limits_{\mathcal{V}} \nabla\cdot\mathbf{E}\,dV, \qquad \text{or} \qquad \int\limits_{\mathcal{V}} \left(\frac{q}{\epsilon_0} - \nabla\cdot\mathbf{E}\right)\,dV = 0,$$

for all spatial regions. Since the integrand is assumed to be continuous, the "zero-integral principle" tells us that it must be identically 0. That is,

$$\frac{q}{\epsilon_0} - \nabla\cdot\mathbf{E} = 0,$$

which give us Coulomb's law. ◇

EXAMPLE 3 Show that II implies II'. That is, $\oint_C \mathbf{E}\cdot\,d\mathbf{r} = 0$ for closed curves implies $\mathbf{curl}\,\mathbf{E} = \mathbf{0}$.

SOLUTION By Stokes' theorem, for any orientable surface $\mathcal{S}$ bounded by a closed curve,

$$\int_{\mathcal{S}} (\mathbf{curl}\, \mathbf{E}) \cdot \mathbf{n}\, dS = 0$$

The zero-integral principle implies that $(\mathbf{curl}\, \mathbf{E}) \cdot \mathbf{n} = 0$ at each point on the surface. Choosing $\mathcal{S}$ such that $\mathbf{n}$ is parallel to $\mathbf{curl}\, \mathbf{E}$ (if $\mathbf{curl}\, \mathbf{E}$ is not $\mathbf{0}$), implies that the magnitude of $\mathbf{curl}\, \mathbf{E}$ is 0, hence $\mathbf{curl}\, \mathbf{E}$ is $\mathbf{0}$. $\diamond$

Maxwell, by studying the four equations, I', II', III', IV', deduced that electromagnetic waves travel at the speed of light, and therefore light is an electromagnetic phenomenon. In CIE 23 at the end of this chapter we show how he accomplished this, in one of the greatest creative insights in the history of science.

The exercises present the analogy of the four equations in integral form for the general case where $\mathbf{B}$ and $\mathbf{E}$ vary with time. It is here that $\mathbf{B}$ and $\mathbf{E}$ became tangled with each other; both appearing in the same equation. In this generality they are known as Maxwell's equations, in honor of James Clerk Maxwell (1831-1879), who put them in their final form in 1865.

Mathematics and Electricity

Benjamin Franklin, in his book *Experiments and Observations Made in Philadelphia*, published in 1751, made electricity into a science. For his accomplishments, he was elected a Foreign Associate of the French Academy of Sciences, an honor bestowed on no other American for over a century. In 1865, Maxwell completed the theory that Franklin had begun.

At the time that Newton Published his *Principia* on the gravitational field (1687), electricity and magnetism were the subjects of little scientific study. But the experiments of Franklin, Oersted, Henry, Ampère, Faraday, and others in the eighteenth and early nineteenth centuries gradually built up a mass of information subject to mathematical analysis. All the phenomena could be summarized in four equations, which in their final form appeared in Maxwell's *Treatise on Electricity and Magnetism*, published in 1873. For a fuller treatment, see *The Feynman Lectures on Physics*, vol. 2, Addison-Wesley, Reading, Mass., 1964.

Summary

We stated the four equations that describe electrostatic and magnetic fields that do not vary with time. Then we showed how to use the divergence theorem or Stokes' theorem to translate between their global and local forms. The exercises include the four equations in their general form, where $\mathbf{E}$ and $\mathbf{B}$ vary with time.

EXERCISES for Section 18.9 *Key:* R–routine, M–moderate, C–challenging

1.[R] Obtain II from II'.

2.[R] Obtain III' from III.

3.[R] Obtain III from III'.

4.[R] Obtain IV' from IV.

5.[R] Obtain IV from IV'.

In Exercises 6 to 9 use terms such as "circulation," "flux," "current," and "charge density" to express the given equation in words.

6.[R] I **8.**[R] III

7.[R] II **9.**[R] IV

10.[R] Which of the four laws tell us that an electric current produces a magnetic field?

11.[R] Which of the four laws tells us that a magnetic field produces an electric current?

In this section we assumed that the fields **E** and **B** do not vary in time, that is, $\partial \mathbf{E}/\partial t = \mathbf{0}$ and $\partial \mathbf{B}/\partial t = \mathbf{0}$. The general case, in empty space, where **E** and **B** depend on time, is also described by four equations, which we call $1, 2, 3, 4$. Numbers 1 and 3, do not involve time; they are similar to I' and III'.

1. $\nabla \cdot \mathbf{E} = q/\epsilon_0$
2. $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$
3. $\nabla \cdot \mathbf{B} = 0$
4. $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{d\mathbf{E}}{dt}$

(Here **J** is the current.)

12.[R] Which equation implies that a changing magnetic field creates an electric field?

13.[R] Which equation implies that a changing electrostatic field creates a magnetic field?

14.[R] Show that 2. is equivalent to

$$\oint_C \mathbf{E} \cdot \ dt = -\frac{\partial}{\partial t} \int_{\mathcal{S}} \mathbf{B} \cdot \mathbf{n}\, dS$$

Here, C bounds S. Hint: You may assume that $\frac{\partial}{\partial t} \int_{\mathcal{S}} \mathbf{B} \cdot \mathbf{n}\, dS$ equals $\int_{\mathcal{S}} (\partial B/\partial t) \cdot \mathbf{n}\, dS$.

15.[R] Show that 4. is equivalent to

$$\oint_C \mathbf{B} \cdot \ d\mathbf{r} = \mu_0 \int_{\mathcal{S}} \mathbf{J} \cdot \mathbf{n}\, dS + \mu_0 \epsilon_0 \frac{\partial}{\partial t} \int_{\mathcal{S}} \mathbf{E} \cdot \mathbf{n}\, dS$$

(The circulation of **B** is related to the total current through the surface $\mathcal{S}$ that C bounds and to the rate at which the flux of **E** through $\mathcal{S}$ changes.)

18.S Chapter Summary

The first six sections developed three theorems: Green's Theorem, Gauss' Theorem (also called the Divergence Theorem), and Stokes' Theorem. The final four sections applied them to geometry and to physics and to expressing various functions in terms of non-rectangular coordinate systems. These four sections offer a way to deepen your understanding of the first six.

Name	Mathematical Expression	Physical I
Green's Theorem	$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_{\mathcal{R}} \nabla\cdot\mathbf{F}\,dA$ $\oint_C(-Qdx+Pdy) = \int_{\mathcal{R}}\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)\,dA$ $\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \oint_C \mathbf{F}\cdot d\mathbf{r} = \int_{\mathcal{R}}(\nabla\times\mathbf{F})\cdot\mathbf{k}\,dA$ $\oint_C(Pdx+Qdy) = \int_{\mathbf{R}}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\,dA$	flux of F differen circulation c
Gauss' Theorem (Divergence Theorem)	$\int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{N}\,dS = \int_R \nabla\cdot\mathbf{F}\,dV$	
Stokes' Theorem	$\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_{\mathcal{S}}(\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS$ ($\mathcal{S}$ is a surface bounded by C with $\mathbf{n}$ compatible by orientation of C)	

Green's Theorem can be viewed as the planar version of either the Divergence Theorem or Stokes' Theorem.

Though div $\mathbf{F}$ and **curl** $\mathbf{F}$ were defined in terms of rectangular coordinates, they also have a meaning that is independent of any coordinates. For instance, if $\mathbf{F}$ is a vector field in space, the divergence of $\mathbf{F}$ at a point multiplied by the volume of a small region containing that point approximates the flux of $\mathbf{F}$ across the surface of that small region. More precisely,

$$\operatorname{div}\mathbf{F} \text{ at } P \text{ equals the limit of } \frac{\int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{n}\,ds}{\text{volume of } \mathcal{R}} \text{ as the diameter of } \mathcal{R} \text{ approaches } 0$$

The **curl** of $\mathbf{F}$ at P is a vector, so it's a bit harder to describe physically. Let $\mathbf{n}$ be a unit vector and C a small curve that lies in a plane through P, is perpendicular to $\mathbf{n}$, and surrounds P. Then the scalar component of **curl** $\mathbf{F}$ at P is the direction $\mathbf{n}$ multiplied by the area of the surface bounded by C gives the circulation of $\mathbf{F}$ along C.

A field whose curl is $\mathbf{0}$ is called irrotational. A field whose divergence is 0 is called incompressible (or divergence-free).

Of particular interest are conservative fields. A field $\mathbf{F}$ is conservative if its circulation on a curve depends only on the endpoints of the curve. If the domain of $\mathbf{F}$ is simply connected, $\mathbf{F}$ is conservative if and only if its curl is $\mathbf{0}$. A conservative field is expressible as the gradient of a scalar function.

Among the conservative fields are the symmetric central fields. If, in addition, they are divergence-free, they take a very special form that depends on the dimension of the problem.

Geometry	General Form of Divergence-Free Symmetric Central Fields	Description
$\mathbf{R}^2$ (plane)	$c\frac{\widehat{\mathbf{r}}}{r}$	inverse radial
$\mathbf{R}^3$ (space)	$c\frac{\widehat{\mathbf{r}}}{r^2}$	inverse square radial
$\mathbf{R}^n$	$c\frac{\widehat{\mathbf{r}}}{r^{n-1}}$	

In the case where $\mathbf{curl}\,\mathbf{F} = \mathbf{0}$ one can replace an integral $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ by an integral over another curve joining A and B. This is most beneficial when the new line integral is easier to evaluate than the original one. Similarly, in a region where $\nabla \cdot \mathbf{F} = 0$ we can replace an integral $\int_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n}\ dS$over the surface $\mathcal{S}$ with a more convenient integral over a different surface.

In applications in space the most important field is the inverse square central field, $\mathbf{F} = \frac{\widehat{\mathbf{r}}}{r^2}$. The flux of this field over a closed surface that does not enclose the origin is 0, but its flux over a surface that encloses the origin is 4π. If one thinks in terms of steradians, it is clear why the second integral is 4π: the flux of $\widehat{\mathbf{r}}/r^2$ also measures the solid angle subtended by a surface. Also, the first case becomes clear when one distinguishes the two parts of the surface where $\mathbf{n} \cdot \mathbf{r}$ is positive and where it is negative.

EXERCISES for 18.S *Key:* R–routine, M–moderate, C–challenging

1.[R] Match the vector fields given in mathematical symbols (a.-e.) with the written description (1.-5.)

a.	$\mathbf{F}(\mathbf{r})$	1.	an inverse cube central field
b.	$f(\mathbf{r})\widehat{\mathbf{r}}$	2.	a central field (center at origin)
c.	$f(r)\widehat{\mathbf{r}}$	3.	an arbitrary vector field
d.	$\widehat{\mathbf{r}}/r^2$	4.	a symmetric central field (center at origin)
e.	$\mathbf{r}/r^3$	5.	an inverse square central field

NOTE: There is not a one-to-one relation between the two columns.

2.[R] Use Green's theorem to evaluate $\oint_C (xy\ dx + e^x\ dy)$, where C is the curve that goes from $(0,0)$ to $(2,0)$ on the x-axis and returns from $(2,0)$ to $(0,0)$ on the parabola $y = 2x - x^2$.

3.[R] A curve C bounds a region $\mathcal{R}$ of area A.

(a) If $\oint_C \mathbf{F} \cdot d\mathbf{r} = -2$, estimate $\nabla \times \mathbf{F}$ at points in R.

(b) Would you use $\odot$ or $\oplus$ to indicate the curl?

4.[R] A curve C bounds a region $\mathcal{R}$ of area A.

(a) If $\oint_C \mathbf{F} \cdot \mathbf{n}\ ds = -2$, estimate $\nabla \cdot \mathbf{F}$ at points in $\mathcal{R}$.

(b) How did you decide whether $\nabla \cdot \mathbf{F}$ is positive or negative?

5.[R] A field $\mathbf{F}$ is called **uniform** if all its vectors are the same. Let $\mathbf{F}(x,y,z) = 3i$.

(a) Find the flux of $\mathbf{F}$ across each of the six faces of the cube in Figure 18.S.1 of side 3.

(b) Find the total flux of **F** across the surface of the box.

(c) Verify the divergence theorem for this **F**.

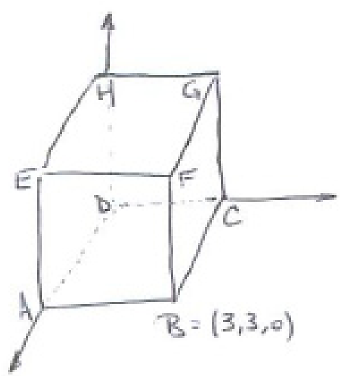


Figure 18.S.1:

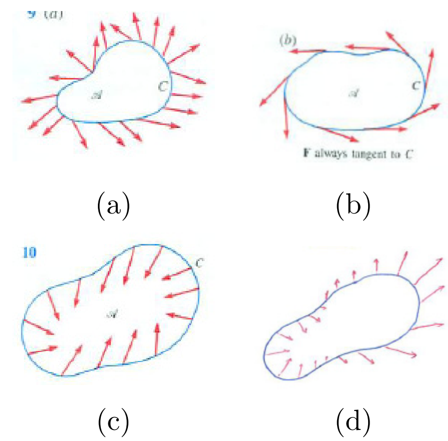
6.[R] Let **F** be the uniform field $\mathbf{F}(x, y, z) = 2\mathbf{i} + 3\mathbf{j} + 0\mathbf{k}$. Repeat Exercise 5 Carry out the preceding exercise for this field.

7.[R] See Exercise 8. Suppose you placed the point at which **E** is evaluated at $(a, 0, 0)$ instead of at $(0, 0, a)$.

(a) What integral in spherical coordinates arises?

(b) Would you like to evaluate it?

In Exercises 8 to 11, **F** is defined on the whole plane but indicated only at points on a curve C bounding a region $\mathcal{R}$. What can be said about $\int_{\mathcal{R}} \nabla \cdot \mathbf{F}\, dA$ in each case?



(a) (b) (c) (d)

Figure 18.S.2:

8.[R] See Figure 18.S.2(a).

9.[R] See Figure 18.S.2(b).

10.[R] See Figure 18.S.2(c).

11.[R] See Figure 18.S.2(d).

Exercises 12 to 15, **F** concern the same **F** as in Exercises 8 to 11. What can be said about $\int_{\mathcal{S}} \nabla \times \mathbf{F}\, dA$ in each case?

12.[R] See Figure 18.S.2(a).

13.[R] See Figure 18.S.2(b).

14.[R] See Figure 18.S.2(c).

15.[R] See Figure 18.S.2(d).

16.[R] Let C be the circle of radius 1 with center $(0, 0)$.

(a) What does Green's theorem say about the line integral

$$\oint_C \left((x^2 - y^3)\, dx + (y^2 + x^3)\, dy \right)?$$

(b) Use Green's theorem to evaluate the integral in (a).

(c) Evaluate the integral in (a) directly.

17.[M] Let $\mathbf{F}(x, y) = (x + y)\mathbf{i} + x^2\mathbf{j}$ and let C be the counterclockwise path around the triangle whose vertices are $(0, 0)$, $(1, 1)$, and $(-1, 1)$.

(a) Use the planar divergence theorem to evaluate $\int_C \mathbf{F} \cdot \mathbf{n} ds$, where $\mathbf{n}$ is the outward unit normal.

(b) Evaluate the line integral in (a) directly.

18.[M] Let b and c be positive numbers and $\mathcal{S}$ the "infinite rectangle" parallel to the xy-plane, consisting of the points (x, y, c) such that $0 \leq x \leq b$ and $b \geq 0$.

(a) If b were replaced by ∞, what is the solid angle $\mathcal{S}$ subtends at the origin? Hint: No integration is needed.

(b) Find the solid angle subtended by $\mathcal{S}$ when b is finite. Hint: See Exercise 94.

(c) Is the limit of your answer in (b) as $b \to \infty$ the same as your answer in (a)? Hint: It should be!

19.[M] Look back at the Fundamental Theorem of Calculus (Section 6.4), Green's Theorem (Section 18.2), the Divergence Theorem (Section 18.6), and Stokes' Theorem (Section 18.4). What single theme runs through all of them?

Calculus is Everywhere # 23

How Maxwell Did It

In a letter to his cousin, Charles Cay, dated January 5, 1965, Maxwell wrote:

> I have also a paper afloat containing an electromagnetic theory of light, which, till I am convinced to the contrary, I hold to be great guns. [Everitt, F., *James Clerk Maxwell: a force for physics*, Physics World, Dec 2006, `http://physicsworld.com/cws/article/print/26527`]

It indeed was "great guns," for out of his theory has come countless inventions, such as television, cell phones, and remote garage door openers. In a dazzling feat of imagination, Maxwell predicted that electrical phenomena create waves, that light is one such phenomenon, and that the waves travel at the speed of light, in a vacuum.

In this section we will see how those predictions came out of the four equations (I'), (II'), (III'), and (IV') in Section 18.9.

First, we take a closer look at the dimensions of the constants ε_0 and μ_0 that appear in (IV'),

$$\frac{1}{\mu_0\varepsilon_0}\nabla \times \mathbf{B} = \frac{\mathcal{J}}{\varepsilon_0}.$$

The constant ε_0 makes its appearance in the equation

$$\text{Force} = F = \frac{1}{4\pi\varepsilon_0}\frac{qq_0}{r^2}. \tag{C.23.1}$$

Since the force F is "mass times acceleration" its dimensions are

$$\text{mass} \cdot \frac{\text{length}}{\text{time}^2},$$

or, in symbols

$$m\frac{L}{T^2}.$$

The number 4π is a pure number, without any physical dimension.

The quantity qq_0 has the dimensions of "charge squared," q^2, and R^2 has dimensions L^2, where L denotes length.

Solving (C.23.1) for ε_0, we find the dimensions of ε_0. Since

$$\varepsilon_0 = \frac{q^2}{4\pi F r^2},$$

its dimensions are

$$\left(\frac{T^2}{mL}\right)\left(\frac{q^2}{L^2}\right) = \frac{T^2q^2}{mL^3}.$$

To figure out the dimensions of μ_0, we will use its appearance in calculating the force between two wires of length L each carrying a current I in the same direction and separated by a distance R. (Each generates a magnetic field that draws the other towards it.) The equation that describes that force is

$$\mu_0 = \frac{2\pi RF}{I^2L}.$$

Since R has the dimensions of length L and F has dimensions mL/T^2, the numerator has dimensions mL^2/T^2. The current I is "charge q per second," so I^2 has dimensions q^2/T^2. The dimension of the denominator is, therefore,

$$\frac{q^2L}{T^2}.$$

Hence μ_0 has the dimension

$$\frac{mL^2}{T^2}\cdot\frac{T^2}{q^2L} = \frac{mL}{q^2}.$$

The dimension of the product $\mu_0\varepsilon_0$ is therefore

$$\frac{mL}{q^2}\cdot\frac{T^2q^2}{mL^3} = \frac{T^2}{L^2}.$$

The dimension of $1/\mu_0\varepsilon_0$, the same as the square of speed. In short, $1/\sqrt{\mu_0\varepsilon_0}$ has the dimension of speed, "length divided by time."

Now we are ready to do the calculations leading to the prediction of waves traveling at the speed of light. We will use the equations (I'), (II'), (III'), and (IV'), as stated on page 1378, where the fields **B** and **E** vary with time. However, we assume there is no current, so $\mathcal{J} = \mathbf{0}$. We also assume that there is no charge q.

Recall the equation (IV')

$$\nabla \times \mathbf{B} = \mu_0\varepsilon_0\frac{\partial \mathbf{E}}{\partial t}.$$

Differentiating this equation with respect to time t we obtain

$$\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) = \mu_0\varepsilon_0\frac{\partial^2 \mathbf{E}}{\partial t^2}. \tag{C.23.2}$$

As is easy to check, the operator $\frac{\partial}{\partial t}$ can be moved past the $\nabla\times$ to operate directly on **B**. Thus (C.23.2) becomes

$$\nabla \times \frac{\partial \mathbf{B}}{\partial t} = \mu_0\varepsilon_0\frac{\partial^2 \mathbf{E}}{\partial t^2}. \tag{C.23.3}$$

Recall the equation (II')

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

Taking the curl of both sides of this equation leads to

$$\nabla \left(-\nabla \times \mathbf{E}\right) = \nabla \times \frac{\partial \mathbf{B}}{\partial t}. \tag{C.23.4}$$

Combining (C.23.3) and (C.23.4) gives us an equation that involves $\mathbf{E}$ alone:

$$\nabla \times \left(-\nabla \times \mathbf{E}\right) = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}. \tag{C.23.5}$$

An identity concerning "the curl of the curl," which tells us that

$$\nabla \times \left(\nabla \times \mathbf{E}\right) = \nabla \left(\nabla \cdot \mathbf{E}\right) - \left(\nabla \cdot \nabla\right) \mathbf{E}. \tag{C.23.6}$$

But $\nabla \cdot \mathbf{E} = 0$ is one of the four assumptions, namely (I), on the electromagnetic fields. By (C.23.5) and (C.23.6), we arrive at

$$\begin{aligned} \left(\nabla \cdot \nabla\right) \mathbf{E} &= \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \\ \text{or} \qquad \frac{\partial^2 \mathbf{E}}{\partial t^2} - \frac{1}{\mu_0 \varepsilon_0} \nabla^2 \mathbf{E} &= \mathbf{0}. \end{aligned} \tag{C.23.7}$$

The expression ∇^2 in (C.23.7) is short for

$$\begin{aligned} \nabla \cdot \nabla &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \cdot \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \\ &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \end{aligned} \tag{C.23.8}$$

In $(\nabla \cdot \nabla)\mathbf{E}$ we apply (C.23.8) to each of the three components of $\mathbf{E}$. Thus $\nabla^2 \mathbf{E}$ is a vector. So is $\partial^2 \mathbf{E}/\partial t^2$ and (C.23.8) makes sense.

For the sake of simplicity, consider the case in which $\mathbf{E}$ has only an x-component, which depends only on x and t, $\mathbf{E}(x, y, z, t) = E(x, t)\mathbf{i}$, where E is a scalar function. Then (C.23.8) becomes

$$\frac{\partial^2}{\partial t^2} E(x,t)\mathbf{i} - \frac{1}{\mu_0 \varepsilon_0} \left(\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2}\right) \mathbf{i} = \mathbf{0},$$

from which it follows

$$\frac{\partial^2}{\partial t^2} E(x,t) - \frac{1}{\mu_0 \varepsilon_0} \frac{\partial^2 E}{\partial x^2} = 0. \tag{C.23.9}$$

Multiply (C.23.9) by $-\mu_0\varepsilon_0$ to obtain

$$\frac{\partial^2 E}{\partial x^2} - \mu_0\varepsilon_0 \frac{\partial^2 E}{\partial t^2} = 0.$$

This looks like the wave equation (see (16.3.11) on page 1101). The solutions are waves traveling with speed $1/\sqrt{\mu_0\varepsilon_0}$.

Maxwell then compares $\sqrt{\mu_0\varepsilon_0}$ with the velocity of light:

> In the following table, the principal results of direct observation of the velocity of light, are compared with the principal results of the comparison of electrical units $(1/\sqrt{\mu_0 v_0})$.

Velocity of light (meters per second)		Ratio of electrical units	
Fizeau	314,000,000	Weber	310,740,000
Sun's Parallax	308,000,000	Maxwell	288,000,000
Foucault	298,360,000	Thomson	282,000,000

Table C.23.1:

> It is magnificent that the velocity of light and the ratio of the units are quantities of the same order of magnitude. Neither of them can be said to be determined as yet with such a degree of accuracy as to enable us to assert that the one is greater or less than the other. It is to be hoped that, by further experiment, the relation between the magnitude of the two quantities may be more accurately determined.
>
> In the meantime our theory, which asserts that these two quantities are equal, and assigns a physical reason for this equality, is certainly not contradicted by the comparison of these results such as they are. [reference?]

On this basis Maxwell concluded that light is an "electromagnetic disturbance" and predicted the existence of other electromagnetic waves. In 1887, eight years after Maxwell's death, Heinrich Hertz produced the predicted waves, whose frequency placed them outside what the eye can see.

By 1890 experiments had confirmed Maxwell's conjecture. First of all, experiments gave the velocity of light as 299,766,000 meters per second and $\sqrt{1/\mu_0\varepsilon_0}$ as 299,550,000 meters per second.

Newton, in his *Principia* of 1687 related gravity on earth with gravity in the heavens. Benjamin Franklin, with his kite experiments showed that lightning was simply an electric phenomenon. From then through the early nineteenth century, Faraday, ???, ...showed that electricity and magnetism were inseparable. Then Maxwell joined them both to light. Einstein, in 1905(?), also by a mathematical argument, hypothesized that mass and energy were related, by his equation $E = mc^2$.

Calculus is Everywhere # 24

Heating and Cooling

Engineers who design a car radiator or a home air conditioner are interested in the distribution of temperature of a fin attached to a tube. We present one of the mathematical tools they use. Incidentally, the example shows how Green's Theorem is applied in practice.

A plane region $\mathcal{A}$ with boundary curve C is occupied by a sheet of metal. By various heating and cooling devices, the temperature along the border is held constant, independent of time. Assume that the temperature in $\mathcal{A}$ eventually stabilizes. This steady-state temperature at point P in $\mathcal{A}$ is denoted $T(P)$. What does that imply about the function $T(x,y)$?

First of all, heat tends to flow "from high to low temperatures," that is, in the direction of $-\nabla T$. According to Fourier's law, flow is proportional to the conductivity of the material k (a positive constant) and the magnitude of the gradient $\|\nabla T\|$. Thus

$$\oint_C (-k\nabla T)\cdot \mathbf{n}ds$$

measures the rate of heat loss across C.

Since the temperature in the metal is at a steady state, the heat in the region bounded by C remains constant. Thus

$$\oint_C (-k\nabla T)\cdot \mathbf{n}ds = 0.$$

Now, Green's theorem then tells us that

$$\int_{\mathcal{A}} \nabla\cdot(-k\nabla T)dA = 0$$

for any region $\mathcal{A}$ in the metal plate. Since $\nabla\cdot\nabla T$ is the Laplacian of T and k is not 0, we conclude that

$$\int_{\mathcal{A}} \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) dA = 0. \tag{C.24.1}$$

By the "zero integrals" theorem, the integrand must be 0 throughout $\mathcal{A}$,

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0.$$

This is an important step, since it reduces the study of the temperature distribution to solving a partial differential equation.

The expression

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2},$$

which is $\nabla \cdot \nabla T$, the divergence of the gradient of T, is called the **Laplacian** of T. If T is a function of x, y, and z, then its Laplacian has one more summand, $\partial^2 T/\partial z^2$. However, the vector notation remains the same, $\nabla \cdot \nabla T$. Even more compactly, it is often reduced to $\nabla^2 T$. Note that in spite of the vector notation, the Laplacian of a scalar field is again a scalar field. A function whose Laplacian is 0 is called "harmonic."

EXERCISES

Summary of Calculus III