Sister Nibedita Government General Degree College for Girls

Department of Mathematics

1. Separation of Variables:

A fundamental technique for obtaining solutions of linear partial differential equations is the method of separation of variables. This means that we look for particular solutions in the form $u(x,y) = X(x)Y(y)$ and try to obtain differential equations for $X(x)$ and $Y(y)$. These equations will contain a parameter called the separation constant. The function $u(x,y)$ is called a separated solution.

Example 1.1: Solve $u_{xx} - 2u_x + u_y = 0$

Solution: If we let $u(x,y) = X(x)Y(y)$ and substitute in equation, we obtain

$X''Y - 2X'Y + XY' = 0 \Longrightarrow \frac{X''-2X'}{X} = \frac{-Y'}{Y} = k$

Where k is separation constant. These equations may be written in the more standard form

$X'' - 2X' - kX = 0$ (1)

$Y' + kY = 0$ (2)

From (1) we get $X(x) = e^x(A_1 e^{x\sqrt{1+k}} + A_2 e^{-x\sqrt{1+k}})$ and From (2) we get $Y(y) = A_3 e^{-ky}$

where $A_1, A_2 \; and \; A_3$ are constants.

Thus the solution is $u(x,t) == e^{x-ky}(Ae^{x\sqrt{1+k}} + Be^{-x\sqrt{1+k}})$

2. <u>Separated solution of Laplace's equation</u>

Consider a two dimensional Laplace equation in cartesian coordinate: $\nabla^2 u = u_{xx} + u_{yy} = 0$.

If we let $u(x,y) = X(x)Y(y)$ and substitute in Laplace's equation, we obtain

$$X''(x)Y(y) + X(x)Y''(y) = 0$$

Dividing by $X(x)Y(y)$ (assumed to be nonzero), we obtain

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = k$$

Where k is separation constant. These equations may be written in the more standard form

$$X'' - kX = 0 \qquad (1)$$

$$Y'' + kY = 0 \qquad (2)$$

Three cases arise,

Case 1. If $k > 0$, we write $k = p^2$, where p is real . The general solutions to (1) and (2) are

$X(x) = A_1 e^{px} + A_2 e^{-px}$ and $Y(y) = A_3 \; cos\, py + A_4\, sin\, py$ where A_1, A_2, A_3, A_4 are arbitrary constants. Thus the solution is $u(x,y) = (A_1 e^{px} + A_2 e^{-px})(A_3 \; cos\, py + A_4\, sin\, py)$

Case 2. If $k = 0$, we have the equations $X" = 0, Y" = 0$, for which the general solutions to (1) and (2) are linear functions: $X(x) = A_1 x + A_2 and \; Y(y) = A_3\, y + A_4$ where A_1, A_2, A_3, A_4 are arbitrary constants.

Thus the solution is $u(x,y) = (A_1 x + A_2)(A_3\, y + A_4)$

Case 3. If $k < 0$, we write $k = -p^2$. The general solutions of (1) and (2) are

$X(x) = A_1 \, cos\, px + A_2\, sin\, px$ and $Y(y) = A_3\, e^{py} + A_4 e^{-py}$ where A_1, A_2, A_3, A_4 are arbitrary constants.

Thus the solution is $u(x,y) = (A_1 \, cos\, px + A_2\, sin\, px)(A_3\, e^{py} + A_4 e^{-py})$

Example 2.1: Find the separated solutions of Laplace's equation $u_{xx} + u_{yy} = 0$ in the region

$0 < x < L, y > 0$ that satisfy the boundary conditions $u(0,y) = 0, u(L,y) = 0, u(x,0) = 0.$

Solution: From the discussion in subsection 2 we have the separated solutions of three types.

<u>In the first case</u>, using the BC we must have

$0 = u(0,y) = (A_1 + A_2)(A_3\, cos\, py + A_4\, sin\, py)$, so $A_2 = -A_1,$

i.e. $u(x,y) = 2A_1\, sinh\, px(A_3 cos\, py + A_4\, sin\, py)$

Again, $0 = u(L,y) = 2A_1\, sinh\, pL(A_3 cos\, py + A_4\, sin\, py)$ implies that $A_1 = 0,$

So, in this case only trivial solution $u(x,y) = 0$ is possible that satisfy the boundary conditions.

<u>In the second case</u>, using the BC we must have $0 = u(0,y) = A_2(A_3 y + A_4)$, so $A_2 = 0,$

i.e. $u(x,y) = A_1 x(A_3\, y + A_4)$

Again, $o = u(L,y) = A_1 L(A_3 y + A_4)$, so $A_1 = 0$. Therefore, in this case only trivial solution $u(x,y) = 0$ is possible that satisfy the boundary conditions.

<u>In the third case</u>, using the BC we must have

$0 = u(0,y) = A_1(A_3\, e^{py} + A_4 e^{-py})$, so that $A_1 = 0;$

i.e. $u(x,y) = A_2\, sin\, px(A_3\, e^{py} + A_4 e^{-py})$

Again, $0 = u(\, L, y) = A_2\, sin\, pL\, (A_3\, e^{pL} + A_4 e^{-pL})$ has a nonzero solution if and only if

$\sin pL = 0$, which is satisfied if and only if $pL = n\pi$ $i.e.$ $p = \frac{n\pi}{L}$ for some n = 1,2,3, ... To satisfy the boundary condition $u(x, 0) = 0$, we must have $A_3 + A_4 = O$, i.e. $A_4 = -A_3$

Therefore $u(x, y) = 2A_2A_3\, sin\frac{n\pi x}{L}\sinh\frac{n\pi y}{L}$

Writing $= 2A_2A_3$, the corresponding eigen functions are

$u_n(x, y) = A_n\, sin\frac{n\pi x}{L}\sinh\frac{n\pi y}{L}$ where n = 1,2,3, ...

Thus, by the principle of superposition , we write the solution is of the form

$$u(x, y) = \sum_{n=1}^{\infty} A_n\, sin\frac{n\pi x}{L}\sinh\frac{n\pi y}{L}$$

Example 2.2: Solve $\nabla^2 u = 0,\ 0 \le x \le \pi,\ 0 \le y \le \pi$

Satisfying the initial boundary condition:

$u(0, y) = 0,\ u(\pi, 0) = 0,\ u(x, \pi) = 0$ and $u(x, 0) = \sin^2 x$

Solution: Similar to the Example 2.1, one of the acceptable general solution is

$$u(x, y) = (c_3 \cos px + c_4 \sin px)(c_1 e^{py} + c_2 e^{-py})$$

Using the BC: $u(0, y) = 0 \Rightarrow c_3 = 0$ and $u(x, \pi) = 0 \Rightarrow c_2 = c_1 e^{2p\pi}$ and $u(\pi, 0) = 0 \Rightarrow c_4 \sin p\pi = 0 \Rightarrow p = n$ where $n = 1,2,3 \ldots$

Therefore, the corresponding eigen functions are

$u_n(x, t) = A_n(e^{ny} - e^{2n\pi}e^{-ny}) \sin nx$ where $n = 1,2,3 \ldots$

Thus, by the principle of superposition , we write the solution is of the form

$u(x, y) = \sum_{n=1}^{\infty} A_n(e^{ny} - e^{2n\pi}e^{-ny}) \sin nx$ (1)

Using the condition $\sin^2 x = u(x, 0) = \sum_{n=1}^{\infty} A_n(1 - e^{2n\pi}) \sin nx$

This is half range fourier series expansion of $\sin^2 x$ in $(0, \pi)$ and hence

$A_n = \frac{2}{\pi(1-e^{2n\pi})}\int_0^{\pi} \sin^2 x \sin n\pi\, dx$ (2) **(Calculate A_n)**

Hence the solution of given problem is given by (1) where A_n is given by (2)

3. Separated solution of Heat equation

Consider the equation $u_t = \alpha u_{xx}$

If we let $u(x, t) = X(x)T(t)$ and substitute in Heat equation, we obtain

$$\frac{X''}{X} = \frac{1}{\alpha}\frac{T'}{T} = k, (a\ seperation\ constant)$$

Then we have

$$X''(x) - k\,X(x) = 0 \qquad (1)$$

$$T'(t) - \alpha k T(t) = 0 \qquad (2)$$

Three cases arise,

Case 1. If $k > 0$, we write $k = p^2$, where p is real . The general solutions to (1) and (2) are

$X(x) = A_1 e^{px} + A_2\, e^{-px}$ and $T(t) = A_3\; e^{\alpha p^2 t}$ where A_1, A_2, A_3 are arbitrary constants. Thus the solution is $u(x,t) = \; e^{\alpha p^2 t}(A_1{}' e^{px} + A_2{}'\, e^{-px})$

Case 2. If $k = 0$, we have the equations $X"(x) = 0, T'(t) = 0$, for which the general solutions (1) and (2) are linear functions: $X(x) = A_1 x + A_2 and\; T(t) = A_3$ where A_1, A_2, A_3 are arbitrary constants.

Thus the solution is $u(x,t) = (\, A_1{}' x + A_2{}')$

Case 3. If $k < 0$, we write $k = -p^2$. The general solutions of (1) and (2) are

$X(x) = A_1 \cos px + A_2 \sin px$ and $T(t) = A_3\, e^{-\alpha p^2 t}$ where A_1, A_2, A_3 are arbitrary constants.

Thus the solution is $u(x,t) = e^{-\alpha p^2 t}(A'_1 \cos px + A'_2 \sin px\ )$

Where $A_1' = A_1 A_3\ \ and\ A_2' = A_2 A_3$

Example 3.1: Find the separated solutions $u(x,t)$ of the heat equation $u_{xx} - u_t = 0$ in the region $0 < x < L, t > 0$ that satisfy the boundary conditions $u(0,t) = 0, u(L,t) = 0$.

Solution. From the discussion in subsection 3 we have the separated solutions of three types.

<u>In case 1</u> Using the boundary condition we get $0 = u(0,t) = (A_1' + A_2')e^{p^2 t}$ i.e. $A'_2 = -A_1'$ and $0 = u(L,t) = \; 2A_1'\, e^{p^2 t} \sinh pL$. Therefore $A_1' = A_2' = 0$

So, in this case only trivial solution $u(x,t) = 0$ is possible that satisfy the boundary conditions.

<u>In case 2</u> Using the boundary condition we get $0 = u(0,t) = A_1'.0 + A_2'$ i.e. $A'_2 = 0$ and $0 = u(L,t) = \; A_1' L$. Therefore $A_1' = A_2' = 0$

So, in this case only trivial solution $u(x,t) = 0$ is possible that satisfy the boundary conditions.

<u>In case 3</u> Using the boundary condition we get $0 = u(0,t) = A_1' e^{-p^2 t}$ i.e. $A'_1 = 0$ and

$0 = u(L,t) = \; A_2'\, e^{-p^2 t} \sin pL$. In order to obtain nonzero solution, $\sin pL = 0$, which is satisfied if and only if $pL = n\pi \;\; i.e.\;\; p = \frac{n\pi}{L}$ for some n = 1,2,3, ...

Therefore, the corresponding eigen functions are

$$u_n(x,t) = A_n \exp\left(-\left(\frac{n\pi}{L}\right)^2 t\right) \sin\frac{n\pi}{L} x \qquad \text{where } n = 1,2,3\,...$$

Thus, by the principle of superposition , we write the solution is of the form

Therefore $u(x,t) = \sum_{n=1}^{\infty} A_n \exp\left(-\left(\frac{n\pi}{L}\right)^2 t\right) \sin\frac{n\pi}{L}x,$ where n = 1,2,3, ...

Example 3.2: Solve the one dimensional diffusion equation in the region $0 \le x \le \pi, t \ge 0$ subject to the conditions

(i) $u(x,t)$ remains finite as $t \to \infty$

(ii) $u = 0$, if $x = 0$ and π for all t

(iii) At $t = 0$, $u(x,t) = \begin{cases} x & 0 \le x \le \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} \le x \le \pi \end{cases}$

Solution: From the discussion in subsection 3 we have the separated solutions of three types.

In case 1, the first condition demands that $u(x,t)$ should remain finite as $t \to \infty$ is not satisfied. Therefore we reject this solution.

In case 2, Using the boundary condition we get $0 = u(0,t) = A_1'.0 + A_2'$ i.e. $A'_2 = 0$ and $0 = u(\pi,t) = A_1'\pi$. Therefore $A_1' = A_2' = 0$.

So, in this case only trivial solution $u(x,t) = 0$ is possible that satisfy the boundary conditions. Since, we are looking for a non-trivial solution, we reject this case.

In case 3, the solution is $u(x,t) = e^{-\alpha p^2 t}(A'_1\, cos\, px \,+\, A'_2\, sin\, px\,)$

Using the BC (ii), we have at $0 = u(0,t) = A_1' e^{-\alpha p^2 t}$ implies $A_1' = 0$ and

$0 = u(\pi,0) = \sin p\pi \Rightarrow p = n$, where $n = 1,2,\ldots$

Hence the solution is found to be of the form

$u_n(x,t) = A_n\, e^{-\alpha n^2 t} \sin nx$ where $n = 1,2,\ldots$

Using the principle of superposition, the solution is

$u(x,t) = \sum_{n=1}^{\infty} A_n\, e^{-\alpha n^2 t} \sin nx$

Using the third condition we get $u(x,0) = \sum_{n=1}^{\infty} A_n \sin nx$

Which is a half range Fourier sine series and therefore

$A_n = \frac{2}{\pi}\int_0^{\pi} u(x,0) \sin nx\, dx = \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} x \sin nx\, dx + \int_{\frac{\pi}{2}}^{\pi} (\pi - x) \sin nx\, dx\right]$

$$\Rightarrow A_n = \frac{4}{n^2\pi} \sin(\frac{n\pi}{2})$$

Thus, the required solution is $u(x,t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{e^{-\alpha n^2 t}\sin(\frac{n\pi}{2})}{n^2}\sin nx$

4. Separated solution of Wave Equation

Consider the wave equation: $u_{tt} = c^2 u_{xx}$

If we let $u(x,t) = X(x)T(t)$ and substitute in Heat equation, we obtain

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = k, (a\ seperation\ constant)$$

Then we have

$$X''(x) - k\,X(x) = 0 \qquad (1)$$

$$T''(t) - c^2 kT(t) = 0 \qquad (2)$$

Three cases arise,

Case 1. If $k > 0$, we write $k = p^2$, where p is real . The general solutions to (1) and (2) are

$X(x) = A_1 e^{px} + A_2\, e^{-px}$ and $T(t) = A_3 e^{cpt} + A_4\, e^{-cpt}$ where A_1, A_2, A_3, A_4 are arbitrary constants. Thus the solution is $u(x,t) = (A_1 e^{px} + A_2\, e^{-px})\,(A_3 e^{cpt} + A_4\, e^{-cpt})$

Case 2. If $k = 0$, we have the equations $X"(x) = 0, T''(t) = 0$, for which the general solutions (1) and (2) are linear functions : $X(x) = A_1 x + A_2 and\ T(t) = A_3 t + A_4$ where A_1, A_2, A_3, A_4 are arbitrary constants. Thus the solution is

$u(x,t) = (A_1 x + A_2)\,(A_3 t + A_4)$

Case 3. If $k < 0$, we write $k = -p^2$. The general solutions of (1) and (2) are

$X(x) = A_1\, cos\, px + A_2\, sin\, px$ and $T(t) = A_3\, cos\, cpt + A_4\, sin\, cpt$ where A_1, A_2, A_3, A_4 are arbitrary constants.

Thus the solution is $u(x,t) = (A_3\, cos\, cpt + A_4\, sin\, cpt)(A_1\, cos\, px + A_2\, sin\, px\ )$

Example 4.1: Obtain the solution of the wave equation $u_{tt} = c^2 u_{xx}$ under the following conditions:

(i) $u(0,t) = u(2,t) = 0$

(ii) $u(x,0) = \sin^3 \frac{\pi x}{2}$

(iii) $u_t(x,0) = 0$

Solution: From the discussion in subsection 4 we have the separated solutions of three types.

In case 1, using the BC(i) we get, $A_1 + A_2 = 0$ and $A_1 e^{2p} + A_2 e^{-2p} = 0$

The above two equations possess non-trivial solution iff

$\begin{vmatrix} 1 & 1 \\ e^{2p} & e^{-2p} \end{vmatrix} = 0 \Rightarrow e^{4p} = 1 \ \Rightarrow 4p = 0 \ \Rightarrow p = 0$ which is against the assumption in the case 1. Hence, this solution is not acceptable.

In case 2, using the BC (i) we get $A_1 0 + A_2 = 0$ and $A_1 . 2 + A_2 = 0 \Rightarrow A_1 = A_2 = 0$.

Hence, only a trivial solution is possible. Since, we are looking for a non-trivial solution, we consider the case 3 and the solution in this case is of the form

$$u(x,t) = (A_3 \, cos \, cpt \ + \ A_4 \, sin \, cpt)(A_1 \, cos \, px \ + \ A_2 \, sin \, px \ )$$

Using the condition $u(0,t) = 0$ gives $A_1 = 0$. Also condition (iii) implies $A_4 = 0$. The condition $u(2,t) = 0$ gives $sin \, 2p = 0 \ \Rightarrow p = \frac{n\pi}{2}$ where $n = 1,2,3 \, ...$

Thus the possible solution is $u(x,t) = \sum_{n=1}^{\infty} A_n \, sin \frac{n\pi x}{2} \, cos \frac{n\pi ct}{2}$

Finally using the condition (iii), we obtain $\sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{2} = \sin^3 \frac{\pi x}{2} = \frac{3}{4} \sin \frac{\pi x}{2} - \frac{1}{4} \sin \frac{3\pi x}{2}$

Which gives $A_1 = \frac{3}{4}, \ A_3 = -\frac{1}{4}$, while all other $A_n's$ are zero. Hence, the required solution is

$$u(x,t) = \frac{3}{4} \sin \frac{\pi x}{2} \cos \frac{\pi ct}{2} - \frac{1}{4} \sin \frac{3\pi x}{2} \cos \frac{3\pi ct}{2}$$