1. Separation of Variables:

A fundamental technique for obtaining solutions of linear partial differential equations is the method of separation of variables. This means that we look for particular solutions in the form 
\[ u(x, y) = X(x)Y(y) \]
and try to obtain differential equations for \( X(x) \) and \( Y(y) \). These equations will contain a parameter called the separation constant. The function \( u(x, y) \) is called a separated solution.

Example 1.1: Solve \( u_{xx} - 2u_x + u_y = 0 \)

Solution: If we let \( u(x, y) = X(x)Y(y) \) and substitute in equation, we obtain
\[ X''Y - 2X'Y + XY' = 0 \Rightarrow \frac{X'' - 2X'}{X} = \frac{-Y'}{Y} = k \]

Where \( k \) is separation constant. These equations may be written in the more standard form
\[ X'' - 2X' - kX = 0 \quad (1) \]
\[ Y' + kY = 0 \quad (2) \]

From (1) we get \( X(x) = e^x(A_1 e^{x\sqrt{1+k}} + A_2 e^{-x\sqrt{1+k}}) \) and From (2) we get \( Y(y) = A_3 e^{-ky} \)
where \( A_1, A_2 \) and \( A_3 \) are constants.

Thus the solution is \( u(x,t) = e^{-ky}(Ae^{x\sqrt{1+k}} + Be^{-x\sqrt{1+k}}) \)

2. Separated solution of Laplace's equation

Consider a two dimensional Laplace equation in cartesian coordinate: \( \nabla^2 u = u_{xx} + u_{yy} = 0 \).

If we let \( u(x, y) = X(x)Y(y) \) and substitute in Laplace's equation, we obtain
\[ X''(x)Y(y) + X(x)Y''(y) = 0 \]

Dividing by \( X(x)Y(y) \) (assumed to be nonzero), we obtain
\[ \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = k \]

Where \( k \) is separation constant. These equations may be written in the more standard form
\[ X'' - kX = 0 \quad (1) \]
\[ Y'' + kY = 0 \quad (2) \]
Three cases arise,

**Case 1.** If \( k > 0 \), we write \( k = p^2 \), where \( p \) is real. The general solutions to (1) and (2) are

\[
X(x) = A_1 e^{px} + A_2 e^{-px} \quad \text{and} \quad Y(y) = A_3 \cos py + A_4 \sin py \quad \text{where} \quad A_1, A_2, A_3, A_4 \quad \text{are arbitrary constants. Thus the solution is} \quad u(x,y) = (A_1 e^{px} + A_2 e^{-px})(A_3 \cos py + A_4 \sin py)
\]

**Case 2.** If \( k = 0 \), we have the equations \( X'' = 0, Y'' = 0 \), for which the general solutions to (1) and (2) are linear functions: \( X(x) = A_1 x + A_2 \) and \( Y(y) = A_3 y + A_4 \) where \( A_1, A_2, A_3, A_4 \) are arbitrary constants.

Thus the solution is \( u(x,y) = (A_1 x + A_2)(A_3 y + A_4) \)

**Case 3.** If \( k < 0 \), we write \( k = -p^2 \). The general solutions of (1) and (2) are

\[
X(x) = A_1 \cos px + A_2 \sin px \quad \text{and} \quad Y(y) = A_3 e^{py} + A_4 e^{-py} \quad \text{where} \quad A_1, A_2, A_3, A_4 \quad \text{are arbitrary constants.}
\]

Thus the solution is \( u(x,y) = (A_1 \cos px + A_2 \sin px)(A_3 e^{py} + A_4 e^{-py}) \)

**Example 2.1:** Find the separated solutions of Laplace’s equation \( u_{xx} + u_{yy} = 0 \) in the region \( 0 < x < L, y > 0 \) that satisfy the boundary conditions \( u(0,y) = 0, u(L,y) = 0, u(x,0) = 0 \).

**Solution:** From the discussion in subsection 2 we have the separated solutions of three types.

In the first case, using the BC we must have

\[ 0 = u(0,y) = (A_1 + A_2)(A_3 \cos py + A_4 \sin py), \quad \text{so} \quad A_2 = -A_1, \]

i.e. \( u(x,y) = 2A_1 \sinh px(A_3 \cos py + A_4 \sin py) \)

Again, \( 0 = u(L,y) = 2A_1 \sinh pL(A_3 \cos py + A_4 \sin py) \) implies that \( A_1 = 0 \),

So, in this case only trivial solution \( u(x,y) = 0 \) is possible that satisfy the boundary conditions.

In the second case, using the BC we must have \( 0 = u(0,y) = A_2(A_3 y + A_4), \quad \text{so} \quad A_2 = 0, \)

i.e. \( u(x,y) = A_1 x(A_3 y + A_4) \)

Again, \( 0 = u(L,y) = A_1 L(A_3 y + A_4), \quad \text{so} \quad A_1 = 0. \) Therefore, in this case only trivial solution \( u(x,y) = 0 \) is possible that satisfy the boundary conditions.

In the third case, using the BC we must have

\[ 0 = u(0,y) = A_1(A_3 e^{py} + A_4 e^{-py}), \quad \text{so that} \quad A_1 = 0; \]

i.e. \( u(x,y) = A_2 \sin px(A_3 e^{py} + A_4 e^{-py}) \)

Again, \( 0 = u(L,y) = A_2 \sin pL(A_3 e^{pL} + A_4 e^{-pL}) \) has a nonzero solution if and only if
\[
sin pL = 0, \text{ which is satisfied if and only if } pL = n\pi \quad i.e. \quad p = \frac{n\pi}{L} \text{ for some } n = 1, 2, 3, \ldots. \text{ To satisfy the boundary condition } u(x, 0) = 0, \text{ we must have } A_3 + A_4 = 0, i.e. A_4 = -A_3.
\]

Therefore \( u(x, y) = 2A_2A_3 \sin \frac{n\pi x}{L} \sinh \frac{n\pi y}{L} \)

Writing \( = 2A_2A_3 \), the corresponding eigen functions are

\[ u_n(x, y) = A_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi y}{L} \quad \text{where } n = 1, 2, 3, \ldots \]

Thus, by the principle of superposition, we write the solution is of the form

\[ u(x, y) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi y}{L} \]

**Example 2.2:** Solve \( \nabla^2 u = 0, \quad 0 \leq x \leq \pi, \quad 0 \leq y \leq \pi \)

Satisfying the initial boundary condition:

\[ u(0, y) = 0, \quad u(\pi, y) = 0, \quad u(x, \pi) = 0 \quad \text{and } u(x, 0) = \sin^2 x \]

**Solution:** Similar to the Example 2.1, one of the acceptable general solution is

\[ u(x, y) = (c_3 \cos px + c_4 \sin px)(c_1 e^{py} + c_2 e^{-py}) \]

Using the BC: \( u(0, y) = 0 \Rightarrow c_3 = 0 \) and \( u(x, \pi) = 0 \Rightarrow c_2 = c_4 e^{2\pi x} \) and \( u(\pi, 0) = 0 \Rightarrow c_4 \sin p\pi = 0 \Rightarrow p = n \) where \( n = 1, 2, 3, \ldots \)

Therefore, the corresponding eigen functions are

\[ u_n(x, t) = A_n (e^{ny} - e^{2\pi n e^{-ny}}) \sin nx \quad \text{where } n = 1, 2, 3, \ldots \]

Thus, by the principle of superposition, we write the solution is of the form

\[ u(x, y) = \sum_{n=1}^{\infty} A_n (e^{ny} - e^{2\pi n e^{-ny}}) \sin nx \quad (1) \]

Using the condition \( \sin^2 x = u(x, 0) = \sum_{n=1}^{\infty} A_n (1 - e^{2\pi n}) \sin nx \)

This is half range fourier series expansion of \( \sin^2 x \) in \((0, \pi)\) and hence

\[ A_n = \frac{2}{\pi (1 - e^{2\pi n})} \int_0^{\pi} \sin^2 x \sin nx dx \quad (2) \quad (\text{Calculate } A_n) \]

Hence the solution of given problem is given by (1) where \( A_n \) is given by (2)

3. **Separated solution of Heat equation**

Consider the equation \( u_t = \alpha u_{xx} \)

If we let \( u(x, t) = X(x)T(t) \) and substitute in Heat equation, we obtain

\[ \frac{X''}{X} = \frac{1}{\alpha} \frac{T'}{T} = k, \quad (a \text{ separation constant}) \]
Then we have

\[ X''(x) - k X(x) = 0 \]  \hfill (1)

\[ T'(t) - akT(t) = 0 \]  \hfill (2)

Three cases arise,

**Case 1.** If \( k > 0 \), we write \( k = p^2 \), where \( p \) is real. The general solutions to (1) and (2) are

\[ X(x) = A_1 e^{px} + A_2 e^{-px} \text{ and } T(t) = A_3 e^{pt} \]

where \( A_1, A_2, A_3 \) are arbitrary constants. Thus the solution is

\[ u(x, t) = e^{pt} (A_1 e^{px} + A_2 e^{-px}) \]

**Case 2.** If \( k = 0 \), we have the equations

\[ X''(x) = 0, \quad T'(t) = 0 \]

for which the general solutions (1) and (2) are linear functions: \( X(x) = A_1 x + A_2 \) and \( T(t) = A_3 \) where \( A_1, A_2, A_3 \) are arbitrary constants.

Thus the solution is

\[ u(x, t) = (A_1' x + A_2') \]

**Case 3.** If \( k < 0 \), we write \( k = -p^2 \). The general solutions of (1) and (2) are

\[ X(x) = A_1 \cos px + A_2 \sin px \text{ and } T(t) = A_3 e^{-pt} \]

Thus the solution is

\[ u(x, t) = e^{-pt} (A_1' \cos px + A_2' \sin px) \]

Where \( A_1' = A_1 A_3 \) and \( A_2' = A_2 A_3 \)

**Example 3.1:** Find the separated solutions \( u(x, t) \) of the heat equation \( u_{xx} - u_t = 0 \) in the region \( 0 < x < L, t > 0 \) that satisfy the boundary conditions \( u(0, t) = 0, u(L, t) = 0 \).

**Solution.** From the discussion in subsection 3 we have the separated solutions of three types.

**In case 1** Using the boundary condition we get \( 0 = u(0, t) = (A_1' + A_2') e^{pt} \) i.e. \( A_2' = -A_1' \) and

\[ 0 = u(L, t) = 2 A_1' e^{pt} \sinh pL \]

Therefore \( A_1' = A_2' = 0 \)

So, in this case only trivial solution \( u(x, t) = 0 \) is possible that satisfy the boundary conditions.

**In case 2** Using the boundary condition we get \( 0 = u(0, t) = A_1', 0 + A_2' \) i.e. \( A_2' = 0 \) and \( 0 = u(L, t) = A_1' L \)

Therefore \( A_1' = A_2' = 0 \)

So, in this case only trivial solution \( u(x, t) = 0 \) is possible that satisfy the boundary conditions.

**In case 3** Using the boundary condition we get \( 0 = u(0, t) = A_1' e^{-pt} \) i.e. \( A_1' = 0 \) and

\[ 0 = u(L, t) = A_2' e^{-pt} \sin pL \]

In order to obtain nonzero solution, \( \sin pL = 0 \), which is satisfied if and only if

\[ pL = n\pi \text{ i.e. } p = \frac{n\pi}{L} \] for some \( n = 1, 2, 3, ... \)

Therefore, the corresponding eigen functions are

\[ u_n(x, t) = A_n \exp \left( -\left( \frac{n\pi}{L} \right)^2 t \right) \sin \frac{n\pi}{L} x \]

where \( n = 1, 2, 3, ... \)
Thus, by the principle of superposition, we write the solution is of the form

\[ u(x, t) = \sum_{n=1}^{\infty} A_n \exp \left(-\left(\frac{n\pi}{L}\right)^2 t\right) \sin \frac{n\pi}{L} x, \text{ where } n = 1, 2, 3, \ldots \]

**Example 3.2:** Solve the one dimensional diffusion equation in the region \(0 \leq x \leq \pi, t \geq 0\) subject to the conditions

(i) \(u(x, t)\) remains finite as \(t \to \infty\)

(ii) \(u = 0\), if \(x = 0\) and \(\pi\) for all \(t\)

(iii) At \(t = 0\), \(u(x, t) = \begin{cases} x & 0 \leq x \leq \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} \leq x \leq \pi \end{cases}\)

**Solution:** From the discussion in subsection 3 we have the separated solutions of three types.

In case 1, the first condition demands that \(u(x, t)\) should remain finite as \(t \to \infty\) is not satisfied. Therefore we reject this solution.

In case 2, Using the boundary condition we get \(0 = u(0, t) = A_1' \cdot 0 + A_2'\) i.e. \(A_2' = 0\) and \(0 = u(\pi, t) = A_1' \pi\). Therefore \(A_1' = A_2' = 0\).

So, in this case only trivial solution \(u(x, t) = 0\) is possible that satisfy the boundary conditions. Since, we are looking for a non-trivial solution, we reject this case.

In case 3, the solution is \(u(x, t) = e^{-ap^2t}(A_1' \cos px + A_2' \sin px)\)

Using the BC (ii), we have at \(0 = u(0, t) = A_1' e^{-ap^2t}\) implies \(A_1' = 0\) and \(0 = u(\pi, 0) = \sin p\pi \Rightarrow p = n, \text{ where } n = 1, 2, \ldots\)

Hence the solution is found to be of the form

\[ u_n(x, t) = A_n e^{-an^2t} \sin nx \text{ where } n = 1, 2, \ldots \]

Using the principle of superposition, the solution is

\[ u(x, t) = \sum_{n=1}^{\infty} A_n e^{-an^2t} \sin nx \]

Using the third condition we get \(u(x, 0) = \sum_{n=1}^{\infty} A_n \sin nx\)

Which is a half range Fourier sine series and therefore

\[ A_n = \frac{2}{\pi} \int_0^{\pi} u(x, 0) \sin nx \, dx = \frac{2}{\pi} \left[ \int_0^{\pi/2} x \sin nx \, dx + \int_{\pi/2}^{\pi} (\pi - x) \sin nx \, dx \right] \]

\[ \Rightarrow A_n = \frac{4}{n^2\pi} \sin \left(\frac{n\pi}{2}\right) \]
4. **Separated solution of Wave Equation**

Consider the wave equation: \( u_{tt} = c^2 u_{xx} \)

If we let \( u(x,t) = X(x)T(t) \) and substitute in Heat equation, we obtain

\[
\frac{X''}{X} = \frac{1}{c^2} \frac{T''}{T} = k, \text{(a separation constant)}
\]

Then we have

\[
X''(x) - kX(x) = 0 \quad (1)
\]

\[
T''(t) - c^2 kT(t) = 0 \quad (2)
\]

Three cases arise,

**Case 1.** If \( k > 0 \), we write \( k = p^2 \), where \( p \) is real. The general solutions to (1) and (2) are

\[
X(x) = A_1 e^{px} + A_2 e^{-px} \quad \text{and} \quad T(t) = A_3 e^{c pt} + A_4 e^{-c pt}
\]

where \( A_1, A_2, A_3, A_4 \) are arbitrary constants. Thus the solution is

\[
u(x,t) = (A_1 x + A_2) (A_3 t + A_4)
\]

**Case 2.** If \( k = 0 \), we have the equations \( X''(x) = 0, T''(t) = 0 \), for which the general solutions (1) and (2) are linear functions: \( X(x) = A_1 x + A_2 \) and \( T(t) = A_3 t + A_4 \) where \( A_1, A_2, A_3, A_4 \) are arbitrary constants. Thus the solution is

\[
u(x,t) = (A_1 x + A_2) (A_3 t + A_4)
\]

**Case 3.** If \( k < 0 \), we write \( k = -p^2 \). The general solutions of (1) and (2) are

\[
X(x) = A_1 \cos px + A_2 \sin px \quad \text{and} \quad T(t) = A_3 \cos c pt + A_4 \sin c pt
\]

where \( A_1, A_2, A_3, A_4 \) are arbitrary constants.

Thus the solution is

\[
u(x,t) = (A_3 \cos c pt + A_4 \sin c pt)(A_1 \cos px + A_2 \sin px)
\]

**Example 4.1:** Obtain the solution of the wave equation \( u_{tt} = c^2 u_{xx} \) under the following conditions:

(i) \( u(0,t) = u(2,t) = 0 \)

(ii) \( u(x,0) = \sin^3 \frac{\pi x}{2} \)

(iii) \( u_t(x,0) = 0 \)

**Solution:** From the discussion in subsection 4 we have the separated solutions of three types.
In case 1, using the BC(i) we get, \( A_1 + A_2 = 0 \) and \( A_1 e^{2p} + A_2 e^{-2p} = 0 \)

The above two equations possess non-trivial solution iff

\[
\begin{vmatrix}
1 & 1 \\
e^{2p} & e^{-2p}
\end{vmatrix} = 0 \Rightarrow e^{4p} = 1 \Rightarrow 4p = 0 \Rightarrow p = 0 \text{ which is against the assumption in the case 1.}
\]

Hence, this solution is not acceptable.

In case 2, using the BC (i) we get \( A_1 0 + A_2 = 0 \) and \( A_1 2 + A_2 = 0 \Rightarrow A_1 = A_2 = 0 \).

Hence, only a trivial solution is possible. Since, we are looking for a non-trivial solution, we consider the case 3 and the solution in this case is of the form

\[
\begin{align*}
u(x, t) &= (A_3 \cos c \text{pt} + A_4 \sin c \text{pt})(A_1 \cos px + A_2 \sin px)
\end{align*}
\]

Using the condition \( u(0, t) = 0 \) gives \( A_1 = 0 \). Also condition (iii) implies \( A_4 = 0 \). The condition \( u(2, t) = 0 \) gives \( \sin 2p = 0 \Rightarrow p = \frac{n\pi}{2} \text{ where } n = 1, 2, 3 \ldots \).

Thus the possible solution is

\[
\begin{align*}
u(x, t) &= \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{2} \cos \frac{n\pi ct}{2}
\end{align*}
\]

Finally using the condition (iii), we obtain

\[
\sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{2} = \sin^2 \frac{\pi x}{2} = \frac{3}{4} \sin \frac{\pi x}{2} - \frac{1}{4} \sin \frac{3\pi x}{2}
\]

Which gives \( A_1 = \frac{3}{4}, A_3 = -\frac{1}{4} \) while all other \( A_n \)'s are zero. Hence, the required solution is

\[
\begin{align*}
u(x, t) &= \frac{3}{4} \sin \frac{\pi x}{2} \cos \frac{\pi ct}{2} - \frac{1}{4} \sin \frac{3\pi x}{2} \cos \frac{3\pi ct}{2}
\end{align*}
\]